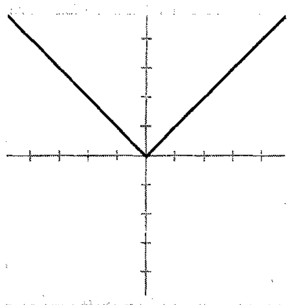
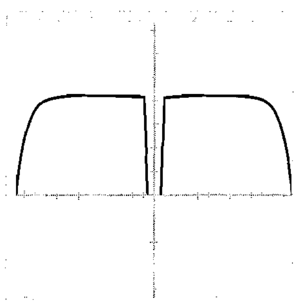


[−5, 5] by [−5, 5]

2.8 The values of $f(x) = |x|$ appear to approach 0 as $x \to 0$.



[−1.3, 1.3] by [−0.5, 0.8]

2.9 The grapher used for this view of $f(x) = (1 - \cos x^6)/x^{12}$ does not have enough precision to produce a correct graph near $x = 0$. You may find it interesting to ZOOM-IN around the origin and the "ends" of the graph. Another interesting view occurs in a $[-0.2, 0.2]$ by $[0.3, 0.7]$ viewing window. In Chapter 7 we will confirm analytically that $\lim_{x\to 0} f(x) = 0.5$.

EXAMPLE 9

Show algebraically that $\lim_{x\to 0}|x| = 0$. Support graphically.

Solution We prove that $\lim_{x\to 0}|x| = 0$ by showing that the right-hand and left-hand limits are both 0:

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0. \qquad |x| = x \quad \text{if} \quad x > 0$$

$$\begin{aligned} \lim_{x\to 0^-} |x| &= \lim_{x\to 0^-} (-x) && |x| = -x \quad \text{if} \quad x < 0 \\ &= -\lim_{x\to 0^-} x && \text{A special case of Theorem 1(c) 4} \\ &= -0 \\ &= 0. \end{aligned}$$

The graph of $f(x) = |x|$ in Fig. 2.8 supports $\lim_{x\to 0}|x| = 0$.

EXPLORATION 4

Complete Graphs

Graphers are very useful in mathematics but cannot be used blindly. To use a grapher for information about a function, including clues as to limiting values, it is essential that viewing-window graphs be complete. GRAPH

$$f(x) = \frac{1 - \cos x^6}{x^{12}}$$

in the standard viewing window and ZOOM-IN to investigate its behavior as $x \to 0$. The view (Fig. 2.9) and TRACE may incorrectly suggest that $f(x) = 0$ for x near 0. Compute $f(x)$ for $x = 0.0001$ and a few other values of x near 0. If your grapher suggests misleading information around 0, you should conclude that the grapher is inadequate for f near 0 (as consideration of the denominator of f might suggest) and that analytic mathematics is *essential* for analyzing f in this trouble spot.

Exercises 2.1

Find the limits in Exercises 1–14 by substitution and support with a grapher.

1. $\lim_{x\to 2} 2x$

2. $\lim_{x\to 0} 2x$

3. $\lim_{x\to 1} (3x - 1)$

4. $\lim_{x\to 1/3} (3x - 1)$

5. $\lim_{x\to -1} 3x(2x - 1)$

6. $\lim_{x\to -1} 3x^2(2x - 1)$

7. $\lim_{x\to -2} (x + 3)^{171}$

8. $\lim_{x\to -4} (x + 3)^{1994}$

9. $\lim_{x\to 1} (x^3 + 3x^2 - 2x - 17)$

10. $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

11. $\lim_{x\to -1} \dfrac{x + 3}{x^2 + 3x + 1}$

12. $\lim_{y\to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

13. $\lim_{y\to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

14. $\lim_{x\to -1} \dfrac{x^3 - 5x + 7}{-x^3 + x^2 - x + 1}$

Explain why substitution does not work to find the limits in Exercises 15–18. Find the limits if they exist.

15. $\lim_{x\to -2} \sqrt{x - 2}$

16. $\lim_{x\to 0} \dfrac{1}{x^2}$

17. $\lim_{x\to 0} \dfrac{|x|}{x}$

18. $\lim_{x\to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 19–26, find the limits graphically. Then confirm algebraically.

19. $\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}$ **20.** $\lim\limits_{x\to -5}\dfrac{x^2+3x-10}{x+5}$

21. $\lim\limits_{t\to 1}\dfrac{t^2-3t+2}{t^2-1}$ **22.** $\lim\limits_{t\to 2}\dfrac{t^2-3t+2}{t^2-4}$

23. $\lim\limits_{x\to 2}\dfrac{2x-4}{x^3-2x^2}$ **24.** $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}$

25. $\lim\limits_{x\to 0}\dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$ **26.** $\lim\limits_{x\to 0}\dfrac{(2+x)^3-8}{x}$

Investigate $\lim_{x\to 0}f(x)$ in Exercises 27–32 by making tables of values.

a)

x	-0.1	-0.01	-0.001	-0.0001	...
$f(x)$	?	?	?	?	...

b)

x	0.1	0.01	0.001	0.0001	...
$f(x)$	?	?	?	?	...

On the basis of the tables, state what you believe the limit to be. Support graphically.

27. $f(x)=x\sin\dfrac{1}{x}$ **28.** $f(x)=\dfrac{1}{x}\sin x$

29. $f(x)=\sin\dfrac{1}{x}$ **30.** $f(x)=\dfrac{10^x-1}{x}$

31. $f(x)=\dfrac{2^x-1}{x}$ **32.** $f(x)=x\sin(\ln|x|)$

33. We have seen that there is little difference in the value of an investment when interest is compounded a few times or many times during a year. Use your grapher and Eq. (1). Find the limit on the value of a \$100 investment as the number of compounding periods within a year increases. Does frequent compounding offer any real advantage to an investor?

34. Set $[0, 1000]$ by $[106, 106.5]$ view dimensions. For each value of i, GRAPH $f(x)=100(1+i/x)^x$ and $g(x)=100e^i$ in the same window. Where x is about 365, ZOOM-IN enough on both graphs so that you can see that the graphs are different. Find the difference between $f(365)$ and $g(365)$. Instead of \$100, how much money would you have to invest for each value of i to have a "real" difference that you could pocket?

a) $i=0.06$ **b)** $i=0.08$

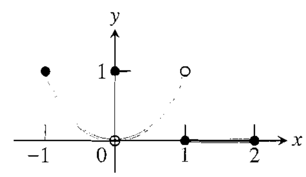
35. GRAPH $f(x)=[(x+2)^2-4]/x$. Magnify the graph around the point $(0,4)$ as much as possible. What do you observe? Confirm your observation algebraically.

36. GRAPH $f(x)=[(x+1)^3-1]/x$. Magnify the graph around point $(0,3)$ as much as possible. What do you observe? Confirm your observation algebraically.

37. GRAPH $f(x)=(x^3-1)/(x-2)$. Use narrow, tall viewing windows to magnify the graph near $x=2$. What do you notice about the values of f as x approaches 2 from the right? From the left? How could you have reached similar conclusions without using a graphing utility? (A *narrow, tall viewing window* is one with a small horizontal view dimension compared to a large vertical view dimension such as $[2, 2.1]$ by $[-2000, 2000]$ to the right of $x=2$, or $[1.99, 2]$ by $[-2000, 2000]$ to the left of $x=2$.)

38. GRAPH $f(x)=\dfrac{1-x^3}{x-2}$. Use tall, narrow viewing windows (see Exercise 37) to magnify the graph near $x=2$. What do you notice about the values of f as x approaches 2 from the right? From the left? How could you have reached similar conclusions without using a graphing utility?

39. Which of the following statements are true of the function $y=f(x)$ graphed here?



a) $\lim\limits_{x\to -1^+}f(x)=1$ **b)** $\lim\limits_{x\to 0^-}f(x)=0$

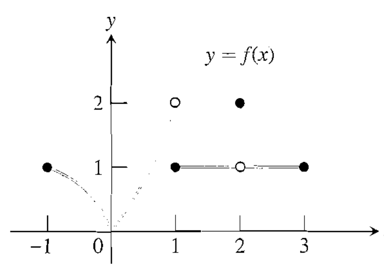
c) $\lim\limits_{x\to 0^-}f(x)=1$ **d)** $\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^+}f(x)$

e) $\lim\limits_{x\to 0}f(x)$ exists **f)** $\lim\limits_{x\to 0}f(x)=0$

g) $\lim\limits_{x\to 0}f(x)=1$ **h)** $\lim\limits_{x\to 1}f(x)=1$

i) $\lim\limits_{x\to 1}f(x)=0$ **j)** $\lim\limits_{x\to 2^-}f(x)=2$

40. Which of the following statements are true of the function graphed here?



a) $\lim\limits_{x\to -1^+}f(x)=1$ **b)** $\lim\limits_{x\to 2}f(x)$ does not exist.

c) $\lim\limits_{x\to 2}f(x)=2$ **d)** $\lim\limits_{x\to 1^-}f(x)=2$

e) $\lim_{x\to1^+} f(x) = 1$ f) $\lim_{x\to1} f(x)$ does not exist.

g) $\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)$

h) $\lim_{x\to c} f(x)$ exists at every c in $(-1, 1)$.

i) $\lim_{x\to c} f(x)$ exists at every c in $(1, 3)$.

Refer to Exploration 5 in Section 1.3 for how to view the piecewise-defined functions in Exercises 41–48 with a grapher.

41. Let $f(x) = \begin{cases} 3-x, & x<2; \\ \dfrac{x}{2}+1, & x>2. \end{cases}$

a) Determine a complete graph of f.
b) Find $\lim_{x\to2^+} f(x)$ and $\lim_{x\to2^-} f(x)$.
c) Does $\lim_{x\to2} f(x)$ exist? If so, what is it? If not, why not?

42. Let $f(x) = \begin{cases} 3-x, & x<2, \\ 2, & x=2, \\ \dfrac{x}{2}, & x>2. \end{cases}$

a) Determine a complete graph of f.
b) Find $\lim_{x\to2^+} f(x)$ and $\lim_{x\to2^-} f(x)$.
c) Does $\lim_{x\to2} f(x)$ exist? If so, what is it? If not, why not?

43. Let $f(x) = \begin{cases} \dfrac{1}{x-1}, & x<1, \\ x^3-2x+5, & x\ge1. \end{cases}$

a) Determine a complete graph of f.
b) Find $\lim_{x\to1^+} f(x)$ and $\lim_{x\to1^-} f(x)$.
c) Does $\lim_{x\to1} f(x)$ exist? If so, what is it? If not, why not?

44. Let $f(x) = \begin{cases} \dfrac{1}{2-x}, & x<2, \\ 5-x^2, & x\ge2. \end{cases}$

a) Determine a complete graph of f.
b) Find $\lim_{x\to2^+} f(x)$ and $\lim_{x\to2^-} f(x)$.
c) Does $\lim_{x\to2} f(x)$ exist? If so, what is it? If not, why not?

45. Let $f(x) = \begin{cases} a-x^2, & x<2, \\ x^2+5x-3, & x\ge2. \end{cases}$
For what values of a does $\lim_{x\to2} f(x)$ exist?

46. Let $f(x) = \begin{cases} x^3-4x, & x<-1, \\ 2x+a, & x\ge-1. \end{cases}$
For what values of a does $\lim_{x\to-1} f(x)$ exist?

47. a) Determine a complete graph of $f(x) = \begin{cases} x^3, & x\ne1, \\ 0, & x=1. \end{cases}$

b) Find $\lim_{x\to1^-} f(x)$ and $\lim_{x\to1^+} f(x)$.
c) Does $\lim_{x\to1} f(x)$ exist? If so, what is it? If not, why not?

48. a) Determine a complete graph of $f(x) = \begin{cases} 1-x^2, & x\ne1, \\ 2 & x=1. \end{cases}$

b) Find $\lim_{x\to1^+} f(x)$ and $\lim_{x\to1^-} f(x)$.
c) Does $\lim_{x\to1} f(x)$ exist? If so, what is it? If not, why not?

Determine a complete graph of the two functions in Exercises 49 and 50. Then answer these questions.

a) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?
b) At what points does only the left-hand limit exist?
c) At what points does only the right-hand limit exist?

49. $f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0\le x<1, \\ 1 & \text{if } 1\le x<2, \\ 2 & \text{if } x=2. \end{cases}$

50. $f(x) = \begin{cases} x & \text{if } -1\le x<0, \text{ or } 0<x\le1, \\ 1 & \text{if } x=0, \\ 0 & \text{if } x<-1, \text{ or } x>1. \end{cases}$

Find the limits of the greatest integer function in Exercises 51–54.

51. $\lim_{x\to0^+} [x]$ **52.** $\lim_{x\to0^-} [x]$

53. $\lim_{x\to0.5} [x]$ **54.** $\lim_{x\to2^-} [x]$

Find the limits in Exercises 55 and 56.

55. $\lim_{x\to0^+} \dfrac{x}{|x|}$ **56.** $\lim_{x\to0^-} \dfrac{x}{|x|}$

Let a be any real number. Find the limits in Exercises 57 and 58.

57. $\lim_{x\to a^+} \dfrac{|x-a|}{x-a}$ **58.** $\lim_{x\to a^-} \dfrac{|x-a|}{x-a}$

59. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = 2$. Find

a) $\lim_{x\to c} f(x)g(x)$ b) $\lim_{x\to c} 2f(x)g(x)$

60. Suppose $\lim_{x\to4} f(x) = 0$ and $\lim_{x\to4} g(x) = 3$. Find

a) $\lim_{x\to4} (g(x)+3)$ b) $\lim_{x\to4} xf(x)$

c) $\lim_{x\to4} g^2(x)$ d) $\lim_{x\to4} \dfrac{g(x)}{f(x)-1}$

61. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a) $\lim_{x\to b} (f(x)+g(x))$ b) $\lim_{x\to b} f(x)\cdot g(x)$

c) $\lim_{x\to b} 4g(x)$ d) $\lim_{x\to b} f(x)/g(x)$

62. Suppose $\lim_{x\to-2} p(x) = 4$, $\lim_{x\to-2} r(x) = 0$, and $\lim_{x\to-2} s(x) = -3$. Find

a) $\lim_{x\to-2}(p(x)+r(x)+s(x))$
b) $\lim_{x\to-2} p(x)\cdot r(x)\cdot s(x)$

Draw a graph and determine the limits in Exercises 63–72.

63. $\lim_{x\to0} x\sin x$ **64.** $\lim_{x\to0} \dfrac{\sin x}{x}$

65. $\lim_{x\to0} x^2\sin x$ **66.** $\lim_{x\to0} \dfrac{1}{x}\sin\dfrac{1}{x}$

67. $\lim_{x\to 0} x^2 \sin \frac{1}{x}$

68. $\lim_{x\to 0} (1+x)^{3/x}$

69. $\lim_{x\to 0} (1+x)^{4/x}$

70. $\lim_{x\to 1} \frac{\ln(x^2)}{\ln x}$

71. $\lim_{x\to 0} \frac{2^x - 1}{x}$

72. $\lim_{x\to 0} \frac{3^x - 1}{x}$

73. Consider the function $f(x) = (1 - \cos x^6)/x^{12}$ of Exploration 4.

a) Reproduce the graph in Fig. 2.9 in both dot and connected mode. Use TRACE to investigate $f(x)$ for x near 0.

b) ZOOM-IN around $x = 0$, and GRAPH in both dot and connected mode.

c) Discuss your findings.

Find the limits in Exercises 74–76 graphically. Does your grapher suggest incorrect information?

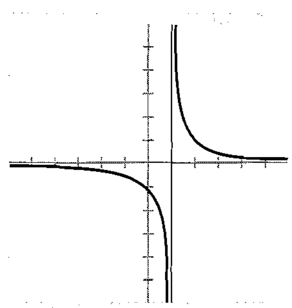
74. $\lim_{x\to 0} \frac{1 - \cos x^{15}}{x^{30}}$

75. $\lim_{x\to 3\pi/2} \frac{(1+\sin x)^{20}}{\left(x - \frac{3\pi}{2}\right)^{40}}$

76. $\lim_{x\to\pi} \frac{(1+\cos x)^{20}}{(x-\pi)^{40}}$

77. *Challenge: limits and geometry.* Let $P = (a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O denote the origin and $(0, b)$ denote the y-intercept of the perpendicular bisector of line segment OP. Evaluate $\lim_{P\to O} b$. Confirm your answer analytically. Support with a grapher.

2.2 Continuous Functions



[-6, 6] by [-6, 6]

2.10 Graphers will sometimes connect the two branches of the graph of $y = 1/(x-1)$, suggesting that the function is defined and continuous at $x = 1$. To avoid "spikes" like this, some graphers allow us to turn off the *connected format* and view graphs in *dot format* to get a better idea of whether the function is really continuous.

Most graphers can plot points (*dot format*). Some can illuminate pixels between plotted points to suggest an unbroken curve (*connected format*). For functions, the connected format basically assumes that outputs vary *continuously* with inputs and do not jump from one value to another without taking on all values in between. (See Fig. 2.10.)

Continuous functions are the functions that we normally use in the equations that describe numerical relations in the world around us. They are the functions we use to find a planet's closest approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions that we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many observable processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as quite a surprise when the physicists of the 1920s discovered that the vibrating atoms in a hydrogen molecule can oscillate only at discrete energy levels, that light comes in particles, and that, when heated, atoms emit light in discrete frequencies and not in continuous spectra.

As a result of these and other discoveries, and because of the heavy use of discrete functions in computer science and statistics, the issue of continuity has become one of practical as well as theoretical importance. As scientists, we need to know when continuity is called for, what it is, and how to test for it.

The Definition of Continuity

A function $y = f(x)$ whose graph can be sketched over any interval of its domain with one continuous motion of the pencil is an example of a **continuous function**. The height of the graph over the interval varies continuously with x. At each interior point of the function's domain, like the point c in