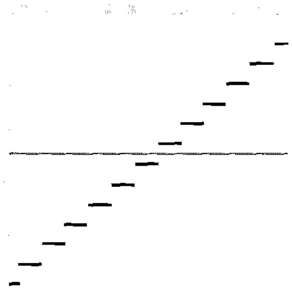
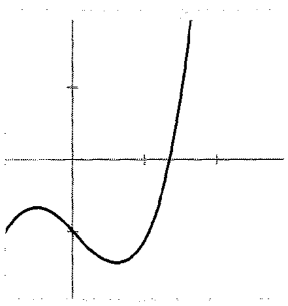


[2, 14] by [−7, 7]

2.22 The graph of a continuous function never "steps" across the $x$-axis the way this step function does. If we know that a function is continuous and its graph appears to cross the $x$-axis, then we know that there indeed is an $x$-intercept.



[−1, 3] by [−2, 2]

2.23 The graph of $f = x^3 - x - 1$ crosses the $x$-axis between $x = 1$ and $x = 2$.

*A consequence for solving equations* Knowing that a function is continuous contributes to knowing that a graph in a viewing window is complete. Suppose that $f(x)$ is continuous at every point of a closed interval $[a, b]$ and that $f(a)$ and $f(b)$ differ in sign. Because zero lies between $f(a)$ and $f(b)$, there is at least one number $c$ between $a$ and $b$ where $f(c) = 0$. In other words, if $f$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval $(a, b)$: the graph of $f$ really has an $x$-intercept where it clearly crosses the $x$-axis. This is the principle that underlies our method of solving equations graphically using ZOOM-IN in Section 1.5. (See Fig. 2.22.)

EXAMPLE 15

Is any real number exactly 1 less than its cube?

**Solution** Any such number must satisfy the equation $x = x^3 - 1$ or, equivalently, $x^3 - x - 1 = 0$. Hence, we are looking for a zero value of the function $f(x) = x^3 - x - 1$. Figure 2.23 suggests that the solution is about 1.3. In Exercise 47, we ask you to find the solution with greater accuracy using ZOOM-IN.

## Concluding Remarks

For any function $y = f(x)$, it is important to distinguish between continuity at $x = c$ and having a limit as $x \to c$. The limit, $\lim_{x \to c} f(x)$, is where the function values are heading as $x \to c$. Continuity is the property of arriving at the point where the $f(x)$ has been heading when $x$ actually gets to $c$. (Someone is home when you get there, so to speak.) If the limit is what you expect as $x \to c$ and the number $f(c)$ is what you get when $x = c$, then the function is continuous at $c$ if you get what you expect.

Finally, remember the test for continuity at a point:

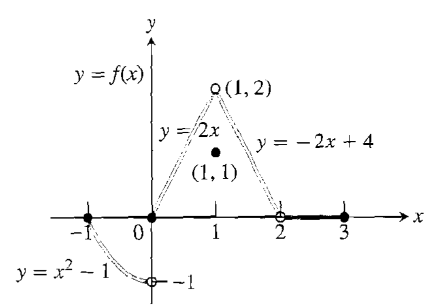
1. Does $f(c)$ exist?
2. Does $\lim_{x \to c} f(c)$ exist?
3. Does $\lim_{x \to c} f(x) = f(c)$?

For $f$ to be continuous at $x = c$, all three answers must be *yes*.

## Exercises 2.2

Exercises 1–6 are about the function $f$ defined as follows and whose graph is shown.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0; \\ 2x, & 0 \le x < 1; \\ 1, & x = 1; \\ -2x + 4, & 1 < x < 2; \\ 0, & 2 < x \le 3. \end{cases}$$



1. a) Does $f(-1)$ exist?
   b) Does $\lim_{x\to -1^+} f(x)$ exist?
   c) Does $\lim_{x\to -1^+} f(x) = f(-1)$?
   d) Is $f$ continuous at $x = -1$?
2. a) Does $f(1)$ exist?
   b) Does $\lim_{x\to 1} f(x)$ exist?
   c) Does $\lim_{x\to 1} f(x) = f(1)$?
   d) Is $f$ continuous at $x = 1$?
3. a) Is $f$ defined at $x = 2$? (Look at the definition of $f$.)
   b) Is $f$ continuous at $x = 2$?
4. At what values of $x$ is $f$ continuous?
5. a) What is the value of $\lim_{x\to 2} f(x)$?
   b) Can a function $g$ be defined to make $g$ a continuous extension of $f$ to the point $x = 2$? If so, give $g$. If not, explain.
6. How should $h$ be defined to make $h$ a continuous extension of $f$ to the point $x = 1$?

At which points are the functions in the following exercises in Section 2.1 continuous?

7. Exercise 39
8. Exercise 40
9. Exercise 41
10. Exercise 42
11. Exercise 43
12. Exercise 44
13. Exercise 47
14. Exercise 48

15. Let $f(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 \le x \le 1, \\ 0, & 1 < x. \end{cases}$
   a) Determine a complete graph of $f$.
   b) At what points is the function continuous?

16. Let $f(x) = \begin{cases} 1, & x < 0, \\ \sqrt{1-x^2}, & 0 \le x \le 1, \\ x-1, & x > 1. \end{cases}$
   a) Determine a complete graph of $f$.
   b) Is $f$ continuous? Explain.

Find the points, if any, at which the functions in Exercises 17–30 are *not* continuous.

17. $y = \dfrac{1}{x-2}$
18. $y = \dfrac{1}{(x+2)^2}$
19. $y = \dfrac{x+1}{x^2-4x+3}$
20. $y = \dfrac{x+3}{x^2-3x-10}$
21. $y = \dfrac{x^3-1}{x^2-1}$
22. $y = \dfrac{1}{x^2+1}$
23. $y = |x-1|$
24. $y = |2x+3|$
25. $y = \dfrac{\cos x}{x}$
26. $y = \dfrac{|x|}{x}$
27. $y = \sqrt{2x+3}$
28. $y = \sqrt[4]{3x-1}$
29. $y = \sqrt[3]{2x-1}$
30. $y = \sqrt[5]{2-x}$

31. The function $f(x)$ is defined by $f(x) = (x^2-1)/(x-1)$ when $x \ne 1$ and by $f(1) = 2$. Is $f$ continuous at $x = 1$? Explain.
32. Define $g(3)$ so that $g(x) = (x^2-9)/(x-3)$ is continuous at $x = 3$.
33. Define $h(2)$ so that $h(x) = (x^2+3x-10)/(x-2)$ is continuous at $x = 2$.
34. Define $f(1)$ so that $f(x) = (x^3-1)/(x^2-1)$ is continuous at $x = 1$.
35. Define $g(4)$ so that $g(x) = (x^2-16)/(x^2-3x-4)$ is continuous at $x = 4$.
36. How should $g$ be defined to make $g$ a continuous extension of $f$ to the point $x = 2$ in Fig. 2.12?
37. What value should be assigned to $a$ to make the function
$$f(x) = \begin{cases} x^2-1, & x < 3, \\ 2ax, & x \ge 3, \end{cases}$$
continuous at $x = 3$? Determine a complete graph of $f$ for this value of $a$.
38. What value should be assigned to $b$ to make the function
$$g(x) = \begin{cases} x^3, & x < 1/2, \\ bx^2, & x \ge 1/2, \end{cases}$$
continuous at $x = 1/2$? Determine a complete graph of $g$ for this value of $b$.

Use the properties of continuous functions in Theorem 5 and the definition of continuous function. Explain why the limits in Exercises 39–42 exist, and give their values. Support your results with a graphing utility.

39. $\lim_{x\to 0} \sec x$
40. $\lim_{x\to 0} \tan x$
41. $\lim_{x\to 0} [(1+\cos x)/2]$
42. $\lim_{x\to 0} \sin\left(\dfrac{\pi}{2}\cos(\tan x)\right)$

In Exercises 43 and 44, investigate the limits using tables of values for $(x, f(x))$ near $x = 0$. On the basis of the tables, state what you believe each limit to be. Support your answer with a graphing utility.

43. $\lim_{x\to 0} \cos\left(1 - \dfrac{\sin x}{x}\right)$
44. $\lim_{x\to 0} \sin\left(\dfrac{1-\cos x}{x}\right)$

45. Let $f(x) = x^3 + 4$. Find $c$ in $[-2, 4]$ for which $f(c) = 2$.
46. Let $f(x) = 2 - x^3$. Find $c$ in $[-2, 2]$ for which $f(c) = 5$.
47. Let $f(x) = x^3 - x - 1$ (see Example 15).
   a) Determine the solution to $f(x) = 0$ with an error of at most $10^{-8}$.
   b) It can be shown that the exact solution in part (a) is
$$\left(\frac{\sqrt{69}}{18} + \frac{1}{2}\right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18}\right)^{1/3}.$$
   Evaluate this answer with a calculator, and compare with the value determined in part (a).

**48.** Let $f(x) = x^3 - 2x + 2$.
**a)** Determine the solution to $f(x) = 0$ with an error of at most $10^{-4}$.
**b)** It can be shown that the exact solution in part (a) is

$$\left(\frac{\sqrt{57}}{9} - 1\right)^{1/3} - \left(\frac{\sqrt{57}}{9} + 1\right)^{1/3}.$$

Evaluate this answer with a calculator, and compare with the value determined in part (a).

**49.** At what values of $x$ (if any) does the function in Fig. 2.12 take on its maximum value? Does the function take on a minimum value? Explain.

**50.** At what values of $x$ (if any) does the function in Fig. 2.23 take on a maximum value? A minimum value?

**51.** Does the function $y = x^2$ have a maximum value on the open interval $-1 < x < 1$? A minimum value? Explain.

**52.** On the closed interval $0 \le x \le 1$, the greatest integer function $y = \lfloor x \rfloor$ takes on a minimum value $m = 0$ and a maximum value $M = 1$. It does so even though it is discontinuous at $x = 1$. Does this violate the Max-Min Theorem? Why?

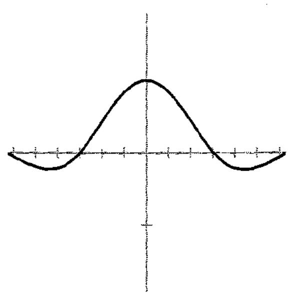
**53.** A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

**54.** Assuming $y = \cos 3x$ to be continuous, show that the equation $\cos 3x = x$ has at least one solution. (*Hint:* Show that the equation $\cos 3x - x = 0$ has at least one solution.)

**55.** Show that $e^{-x} = x$ has at least one solution.

**56.** *Group discussion.* You hear a calculus teacher say, "A one-point gap on a graph is *not* visible." Write a paragraph explaining what you think the teacher means. Explain why you sometimes see a one-point gap on a grapher.

## 2.3 The Sandwich Theorem and $(\sin\theta)/\theta$



$[-2\pi, 2\pi]$ by $[-2, 2]$

**2.24** The graph suggests that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,$$

even though $(\sin\theta)/\theta$ is undefined at $\theta = 0$.

A useful fact in calculus is that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ when $\theta$ is measured in radians. (Unless explicitly stated otherwise, trigonometric arguments will be in radians.) This beautiful and simple result turns out to be the key to measuring the rates at which all trigonometric functions of $\theta$ change their values as $\theta$ changes, as we shall see in Chapter 3.

Figure 2.24 provides support that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$. To confirm it, we introduce and apply a powerful theorem called the Sandwich Theorem.

**THEOREM 1** **The Sandwich Theorem**

Suppose that

$$g(x) \le f(x) \le h(x)$$

for all $x \ne c$ in some interval about $c$ and that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then

$$\lim_{x\to c} f(x) = L.$$

We have included a proof of the Sandwich Theorem in Appendix 2. The idea of this theorem is that if the values of $f$ are sandwiched between the values of two functions that approach $L$, then the values of $f$ also approach $L$ (Fig. 2.25).