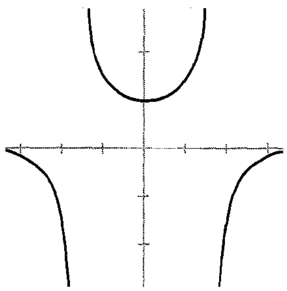
EXAMPLE 9

Find $\lim_{x\to 0}\dfrac{\tan x}{x}$ graphically. Confirm algebraically.

Solution The graph of $f(x) = (\tan x)/x$ (Fig. 2.29) suggests that $\lim_{x\to 0}(\tan x)/x = 1$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{x}\,\frac{1}{\cos x} && \tan x = \frac{\sin x}{\cos x}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\,\lim_{x\to 0}\frac{1}{\cos x} && \text{Property of Limits}\\
&= 1\cdot 1\\
&= 1
\end{aligned}$$



$[-\pi, \pi]$ by $[-3, 3]$

2.29 The graph of $f(x) = (\tan x)/x$ suggests that $f(x) \to 1$ as $x \to 0$.

EXPLORATION 1

Confirm Algebraically

Find each limit graphically. Then confirm algebraically using the limit of $(\sin\theta)/\theta$.

1. $\lim_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$
2. $\lim_{x\to 0}\dfrac{\sin 5x}{\sin 8x}$
3. Make a conjecture about $\lim_{x\to 0}(\sin ax/\sin bx)$. Then prove your conjecture true or find a counterexample to prove it false.

Exercises 2.3

Find the limits in Exercises 1–16 graphically and confirm algebraically.

1. $\lim\limits_{x\to 0}\dfrac{1}{\cos x}$

2. $\lim\limits_{x\to 0}(2\sin x + 3\cos x)$

3. $\lim\limits_{x\to 0}\dfrac{1+\sin x}{1+\cos x}$

4. $\lim\limits_{x\to 0}\dfrac{x^2+1}{1-\sin x}$

5. $\lim\limits_{x\to 0}\dfrac{x}{\sin x}$

6. $\lim\limits_{x\to 0}\dfrac{x}{\tan x}$

7. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

8. $\lim\limits_{x\to 0}\dfrac{x}{\sin 3x}$

9. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{2x}$

10. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$

11. $\lim\limits_{x\to 0}\dfrac{\sin x}{2x^2 - x}$

12. $\lim\limits_{x\to 0}\dfrac{x+\sin x}{x}$

13. $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x}$

14. $\lim\limits_{x\to 0}\dfrac{\tan^2 x}{2x}$

15. $\lim\limits_{x\to 0}\dfrac{3\sin 4x}{\sin 3x}$

16. $\lim\limits_{x\to 0}\dfrac{\tan 5x}{\tan 2x}$

17. Let $f(x) = \dfrac{\tan 3x}{\sin 5x}$.

a) Estimate $\lim_{x\to 0} f(x)$ graphically. (*Hint:* ZOOM-IN around $x = 0$).
b) Compare the function values of f near 0 to the limit found in part (a).
c) Find the exact value of $\lim_{x\to 0} f(x)$ algebraically.

18. Let $f(x) = \dfrac{\cot 3x}{\csc 2x}$.

a) Estimate $\lim_{x\to 0} f(x)$ graphically. (*Hint:* ZOOM-IN around $x = 0$).
b) Compare the function values of f near 0 to the limit found in part (a).
c) Find the exact value of $\lim_{x\to 0} f(x)$ algebraically.

19. *Sandwich Theorem.* The inequality

$$1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1$$

holds when x is measured in radians and $-1 < x < 1$. Use this inequality to calculate $\lim_{x\to 0}(\sin x)/x$. In Chapter 9 we shall see where this inequality comes from.

Sandwich Theorem. As we saw in the proof that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ and again in Exercise 19, we can sometimes use the Sandwich Theorem to find the limit of a fraction whose numerator and denominator both approach zero. Exercises 20–22 contain three other examples in which the inequalities come from infinite series (Chapter 9) and are true for radian values of x close to zero. In each exercise:

a) Apply the Sandwich Theorem graphically by graphing the three functions in the inequality simultaneously.

b) Give the value of $\lim_{x\to 0} f(x)$ suggested by the graphs.

c) Confirm the limit analytically by applying the Sandwich Theorem.

20. $1 - \dfrac{x^2}{6} < f(x) = \dfrac{x \sin x}{2 - 2\cos x} < 1$

21. $\dfrac{1}{2} - \dfrac{x^2}{24} < f(x) = \dfrac{1 - \cos x}{x^2} < \dfrac{1}{2}$

22. $1 < f(x) = \dfrac{\tan x}{x} < 1 + x^2$

23. Show that $\lim_{x\to 0}(\cos x)/x$ does not exist.

24. The area formula $A = (1/2)r^2\theta$ derived in Fig. 2.26 for radian measure has to be changed if the angle is measured in degrees. What should the new formula be?

25. Find $\lim_{\theta\to 0}(\sin\theta)/\theta$ if θ is measured in degrees.

Investigate $\lim_{x\to 0} f(x)$ in Exercises 26–29 by making tables of values. On the basis of the tables, state what you believe the limit to be. Support graphically.

26. $f(x) = \dfrac{\sin 2x}{x}$

27. $f(x) = \dfrac{\tan 3x}{x}$

28. $f(x) = \dfrac{\cos x - 1}{x}$

29. $f(x) = \dfrac{x - \sin x}{x^2}$

30. Give a geometric proof (similar to the one given for $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}$) to show that $\lim_{\theta\to 0}\dfrac{1 - \cos\theta}{\theta} = 0$.

2.4 Limits Involving Infinity

Although there is no real number *infinity*, the word *infinity* is useful for describing how some functions behave when their domains or ranges exceed all bounds. In this section we describe what it means for the values of a function to approach infinity and what it means for a function $f(x)$ to have a limit as x approaches infinity. Our presentation continues to be informal. In Section 2.6 we define the limits involving infinity more precisely.

> **THE SYMBOL ∞**
>
> The symbol ∞, read "infinity," does not represent any real number. We cannot use ∞ in arithmetic in the usual way, but it is convenient to be able to say things like "the limit of $1/x$ as x approaches infinity is 0." Here, "x approaches infinity" means that x *increases without bound.*

Functions with Finite Limits as $x \to \pm\infty$

Our strategy is again the one that we used in Section 2.1. We find the limits of two "basic" functions as $x \to \infty$ and $x \to -\infty$, and then, to find everything else, we use a theorem about limits of algebraic combinations. In Section 2.1 the basic functions were the constant function $y = k$ and the identity function $y = x$. Here, the basic functions are $y = k$ and the reciprocal function $y = 1/x$.