Changing Variables with Substitutions

EXPLORATION BIT

Example 15 suggests a creative way to "see" limits at infinity on your graphing utility. Can you develop the procedure? (We'll show you how later in this book.)

Sometimes a change of variable can turn an unfamiliar expression into one whose limit we know how to find. Here are two examples.

EXAMPLE 15

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{\theta\to 0^+} \sin\theta \qquad \text{Substitute } \theta = 1/x. \text{ Then } \theta \to 0^+ \text{ as } x \to \infty.$$

$$= 0. \qquad \text{Sin } \theta \text{ is continuous.}$$

EXAMPLE 16

Determine the limit graphically. Confirm algebraically using substitution.

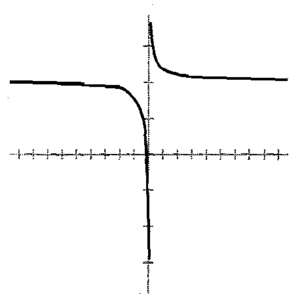
$$\lim_{x\to\pm\infty}\left(1+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)$$

Solution The graph of $f(x) = [1 + (2/x)][\cos(1/x)]$ in Fig. 2.43 suggests that the limit is 1. Algebraically,

$$\lim_{x\to\pm\infty}\left(1+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right) = \lim_{\theta\to 0}(1+2\theta)(\cos\theta) \qquad \text{Substitute } \theta = \frac{1}{x}. \text{ Then } \theta \to 0 \text{ as } x \to \pm\infty$$

$$= \lim_{\theta\to 0}(1+2\theta)\lim_{\theta\to 0}\cos\theta \qquad \text{Product Rule}$$

$$= 1\cdot 1 = 1. \qquad \text{The functions are continuous.}$$



[−100, 100] by [−2, 2]

2.43 The values of

$$f(x) = \left(1+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)$$

approach 1 as $x \to \pm\infty$. Use TRACE to estimate the limit.

Exercises 2.4

Use algebra to find the limits of the functions defined by the expressions in Exercises 1–12 (a) as $x \to \infty$ and (b) as $x \to -\infty$. Support graphically.

1. $\dfrac{2x+3}{5x+7}$
2. $\dfrac{2x^3+7}{x^3-x^2+x+7}$
3. $\dfrac{x+1}{x^2+3}$
4. $\dfrac{3x+7}{x^2-2}$
5. $\dfrac{3x^2-6x}{4x-8}$
6. $\dfrac{x^4}{x^3+1}$
7. $\dfrac{1}{x^3-4x+1}$
8. $\dfrac{10x^5+x^4+31}{x^6}$
9. $\dfrac{-2x^3-2x+3}{3x^3+3x^2-5x}$
10. $\dfrac{-x^4}{x^4-7x^3+7x^2+9}$
11. $\left(\dfrac{-x}{x+1}\right)\left(\dfrac{x^2}{5+x^2}\right)$
12. $\left(\dfrac{2}{x}+1\right)\left(\dfrac{5x^2-1}{x^2}\right)$

Find the limits in Exercises 13–20 by a convincing method of your choice.

13. $\lim_{x\to 2^+}\dfrac{1}{x-2}$
14. $\lim_{x\to 2^-}\dfrac{1}{x-2}$
15. $\lim_{x\to 2^+}\dfrac{x}{x-2}$
16. $\lim_{x\to 2^-}\dfrac{x}{x-2}$
17. $\lim_{x\to -3^+}\dfrac{1}{x+3}$
18. $\lim_{x\to -3^-}\dfrac{1}{x+3}$
19. $\lim_{x\to -3^+}\dfrac{x}{x+3}$
20. $\lim_{x\to -3^-}\dfrac{x}{x+3}$

Find the end behavior asymptote algebraically in Exercises 21–30 and support graphically. Find all vertical asymptotes.

21. $f(x) = \dfrac{x-2}{2x^2+3x-5}$
22. $T(y) = \dfrac{2y+3}{4-y^2}$
23. $g(x) = \dfrac{3x^2-x+5}{x^2-4}$
24. $f(x) = \dfrac{2-3x^2}{5+2x-6x^2}$
25. $f(x) = \dfrac{x^2-2x+3}{x+2}$
26. $f(x) = \dfrac{x^2-3x-7}{x+3}$
27. $g(x) = \dfrac{x^3-2x+1}{x-2}$
28. $g(x) = \dfrac{x^4-2x^2-x+3}{x^2-4}$

29. $f(x) = \frac{x}{x^2+3} - \frac{x^2+2}{1+x-x^2}$

30. $g(x) = \frac{x^2-4}{x+1} + \frac{x}{x^2-3x+2}$

In Exercises 31–34, use graphs to find the limits.

31. a) $\lim_{x\to 2^+} \frac{1}{x^2-4}$ b) $\lim_{x\to 2^-} \frac{1}{x^2-4}$

c) $\lim_{x\to -2^+} \frac{1}{x^2-4}$ d) $\lim_{x\to -2^-} \frac{1}{x^2-4}$

32. a) $\lim_{x\to 1^+} \frac{x}{x^2-1}$ b) $\lim_{x\to 1^-} \frac{x}{x^2-1}$

c) $\lim_{x\to -1^+} \frac{x}{x^2-1}$ d) $\lim_{x\to -1^-} \frac{x}{x^2-1}$

33. a) $\lim_{x\to -2^+} \frac{x^2-1}{2x+4}$ b) $\lim_{x\to -2^-} \frac{x^2-1}{2x+4}$

34. a) $\lim_{x\to 0^+} \left(x^2 + \frac{4}{x}\right)$ b) $\lim_{x\to 0^-} \left(x^2 + \frac{4}{x}\right)$

Find the limits in Exercises 35–42 by a convincing method of your choice.

35. $\lim_{x\to 0^+} \frac{[x]}{x}$ **36.** $\lim_{x\to 0^-} \frac{[x]}{x}$

37. $\lim_{x\to\infty} \frac{|x|}{|x|+1}$ **38.** $\lim_{x\to -\infty} \frac{x}{|x|}$

39. $\lim_{x\to 0^+} \frac{1}{\sin x}$ **40.** $\lim_{x\to 0^-} \frac{1}{\sin x}$

41. $\lim_{x\to (\pi/2)^+} \frac{1}{\cos x}$ **42.** $\lim_{x\to (\pi/2)^-} \frac{1}{\cos x}$

43. Let $f(x) = \begin{cases} \frac{1}{x}, & x < 0, \\ -1, & x \ge 0. \end{cases}$

Find $\lim f(x)$ as $x \to -\infty$, 0^-, 0^+, and ∞.

44. Let $f(x) = \begin{cases} \frac{x-2}{x-1}, & x \le 0, \\ \frac{1}{x^2}, & x > 0. \end{cases}$

Find $\lim f(x)$ as $x \to -\infty$, 0^-, 0^+, and ∞.

Find the limits in Exercises 45–50.

45. $\lim_{x\to\infty} \left(2 + \frac{\sin x}{x}\right)$ **46.** $\lim_{x\to -\infty} \frac{\sin x}{x}$

47. $\lim_{x\to\infty} \left(1 + \cos\frac{1}{x}\right)$ **48.** $\lim_{x\to\infty} x \sin\frac{1}{x}$

49. $\lim_{x\to\infty} \frac{\sin 2x}{x}$ **50.** $\lim_{x\to\infty} \frac{\cos(1/x)}{1+(1/x)}$

Use the Sandwich Theorem for limits at ∞ to find the limits in Exercises 51 and 52.

51. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ if

$$\frac{2x^2}{x^2+1} < f(x) < \frac{2x^2+5}{x^2}.$$

52. *The greatest integer function.* Find $\lim_{x\to\infty}[x]/x$ and $\lim_{x\to -\infty}[x]/x$ given that

$$\frac{x-1}{x} < \frac{[x]}{x} \le 1 \quad (x \ne 0).$$

53. Draw complete graphs of the following functions in the same viewing window: $2x$, $2x^3$, $2x^5$, $2x^7$. Compare their limits as $x \to \pm\infty$ and the steepness of the graphs. How can you distinguish their behavior? Explain.

54. Draw complete graphs of the following functions in the same viewing window: $-3x^2$, $-3x^4$, $-3x^6$, $-3x^8$. Compare their limits as $x \to \pm\infty$ and the steepness of the graphs. How can you distinguish their behavior? Explain.

55. Show that $y = -1/7$ is an end behavior model for the function

$$f(x) = -\frac{x}{7x+4}$$

of Example 11. (*Hint:* Show that $\lim_{x\to\pm\infty} f(x)/(-1/7) = 1$.)

56. Show that $y = 2/3$ is an end behavior model for the function

$$f(x) = \frac{2x^2-x+3}{3x^2+5}$$

of Example 12.

In Exercises 57 and 58, sketch a graph of a function $y = f(x)$ with domain the largest subset of real numbers that satisfies the stated conditions.

57. $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 5^-} f(x) = \infty$,

$\lim_{x\to\infty} f(x) = -1$, $\lim_{x\to -\infty} f(x) = -\infty$

58. $\lim_{x\to 2} f(x) = -1$, $\lim_{x\to 4^+} f(x) = -\infty$,

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 2$

In Exercises 59–62, find the limits (or state that they do not exist) of f, g, and fg as $x \to c$.

59. $f(x) = \frac{1}{x}$, $g(x) = x$, $c = 0$

60. $f(x) = -\frac{2}{x^3}$, $g(x) = 4x^3$, $c = 0$

61. $f(x) = \frac{3}{x-2}$, $g(x) = (x-2)^3$, $c = 2$

62. $f(x) = \frac{5}{(3-x)^4}$, $g(x) = (x-3)^2$, $c = 3$

63. Let $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Give examples to show that $\lim_{x\to c}(fg)$ can be 0, finite and nonzero, or infinite.

64. Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \pm\infty$. Can $\lim_{x\to c}(f \pm g)$ be determined? Explain.

65. Prove Theorem 11.

66. Use graphs to find the limits.

a) $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x}$.

b) $\lim_{x\to\infty} \frac{\ln(x+999)}{\ln x}$. Compare with part (a).

c) $\lim_{x\to\infty} \frac{\ln x^2}{\ln x}$.

d) $\lim_{x\to\infty} \frac{\ln x}{\log x}$.

In Exercises 67–74, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. The behavior of the functions will be explained in Chapter 7.

67. $f(x) = \left(1 + \frac{1}{x}\right)^x$ **68.** $f(x) = \left(1 + \frac{5}{x}\right)^x$

69. $f(x) = \left(1 + \frac{0.07}{x}\right)^x$ **70.** $f(x) = \left(1 + \frac{1}{x}\right)^{1/x}$

71. $f(x) = xe^{-x}$ **72.** $f(x) = x^2e^{-x}$

73. $f(x) = xe^x$ **74.** $f(x) = x^2e^x$

75. Explain why there is no value L (not even $\pm\infty$) for which $\lim_{\theta\to\infty} \sin\theta = L$.

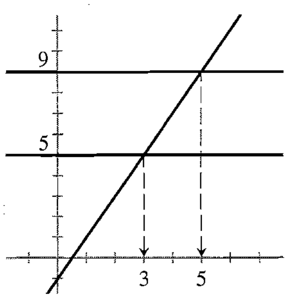
76. *Group discussion.* Discuss the number of horizontal and vertical asymptotes possible for a rational function. Illustrate with examples. Use the function of Example 13 as the example for its type of rational function.

2.5 Controlling Function Outputs: Target Values

We sometimes want the outputs of a function $y = f(x)$ to lie near a particular target value y_0. This need can come about in different ways. A gas-station attendant, asked for $5.00 worth of gas, will try to pump the gas to the nearest cent. A mechanic grinding a 3.385-in. cylinder bore knows to not let the bore vary from this value by more than 0.002 in. A pharmacist making ointments will measure the ingredients to the nearest milligram.

So the question becomes: How accurate do our machines and instruments have to be to keep the outputs within useful bounds? When we express this question with mathematical symbols, we ask: How closely must we control x to keep $y = f(x)$ within an acceptable interval about some particular target value y_0? In this section we use a graphing utility to give us clues to the answers for specific examples. Then we confirm the answers algebraically. In Section 2.6 we use ideas from this section to give a general answer with the formal definition of limit.

Controlling Function Outputs as $x \to c$



[−2, 8] by [−2, 12]

2.44 Keeping x between 3 and 5 will keep $y_1 = 2x - 1$ between $y_2 = 5$ and $y_3 = 9$.

EXPLORATION 1

Aiming at the Target

Controlling a linear function For the *linear function* $y = 2x - 1$, $y = 7$ when $x = 4$. How close must we hold x to $x_0 = 4$ for y to be within 2 units of $y_0 = 7$? In other words, we want to find x values around 4 that give y values between 5 and 9.

The viewing window (Fig. 2.44) shows graphs of $y_1 = 2x - 1$, $y_2 = 5$, and $y_3 = 9$. A TRACE on $y_1 = 2x - 1$ shows that $5 < y < 9$ when $3 < x < 5$. Confirming algebraically, we get

$$3 < x < 5 \Rightarrow 6 < 2x < 10$$
$$\Rightarrow 5 < 2x - 1 < 9.$$