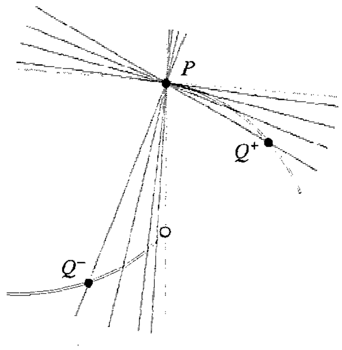
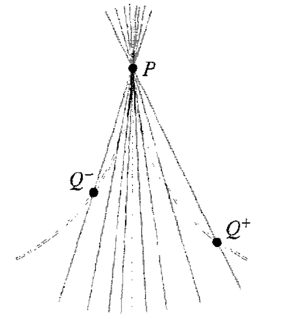




3. A *vertical tangent* (above right) where a graph is vertical "for an instant" as is the graph of $f(x) = -5\sqrt[3]{|x|}$ at $x = 0$. (See Exercise 41.)

Part 2 above illustrates the contrapositive of Theorem 1: If f is not continuous at $x = c$, then $f'(c)$ does not exist. (The contrapositive of an "if P, then Q" theorem is the true statement "if not Q, then not P.") Recall the Continuity Test from Section 2.2. Now, if any part of the test fails at $x = c$, then f is not continuous at that point *and* it is not differentiable there either. Here are the three ways in which the test can fail:

1. $f(c)$ does not exist. *Example:* $f = 1/x$ is not differentiable at $x = 0$.
2. $\lim_{x \to c} f(x)$ does not exist. *Example:* Illustrated in part 2 above.
3. $\lim_{x \to c} f(x) \neq f(c)$. *Example:* Think of $f(c)$ defining a removable discontinuity.

What Functions Are Differentiable?

Most of the functions that we have worked with so far are differentiable. Polynomials are differentiable, as are rational functions and trigonometric functions. Composites of differentiable functions are differentiable, and so are sums, differences, products, powers, and the quotients of differentiable functions, where defined. We will explain all this as the chapter continues.

Exercises 3.2

In Exercises 1–6, use NDER to find an equation for the tangent line to the curve at the given point. Then graph the curve and the tangent line in the same viewing window.

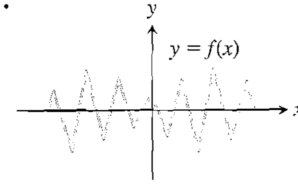
1. $y = x^2 + 1,\ x = 2.$
2. $y = 2x^3 - 5x - 2,\ x = 1.5.$
3. $y = \sqrt{4 - x^2},\ x = -1.$
4. $y = (x - 1)^3 + 1,\ x = 2.5.$
5. $y = \dfrac{x^2 - 4}{x^2 + 1},\ x = 2.$
6. $y = x \sin x,\ x = 2.$

Which of the graphs in Exercises 7–10 suggest a function $y = f(x)$ that is

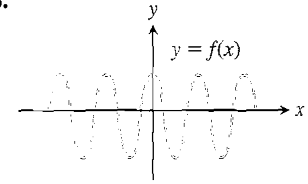
a) Continuous at every point of its domain?
b) Differentiable at every point of its domain?
c) Both (a) and (b)?
d) Neither (a) nor (b)?

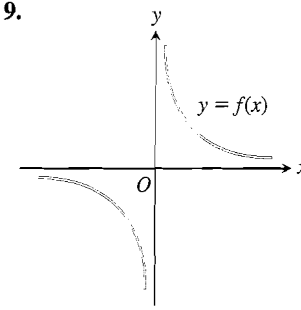
Explain in each case.

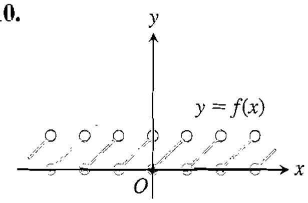
7.



8.



9. 

10. 

Let $D(h) = (f(a+h) - f(a))/h$ and $S(h) = (f(a+h) - f(a-h))/(2h)$ for each function f in Exercises 11–18.

a) For $a = 2$ and $a = 0$, complete a table of values comparing $D(h)$ (difference quotient) and $S(h)$ (symmetric difference quotient) for $h = -0.1, 0.1, -0.01, 0.01, -0.001, 0.001, -0.0001, 0.0001, -10^{-15}, 10^{-15}$.

b) Make a conjecture about the exact value of $f'(a)$, or state that it does not exist. Which of $D(h)$ or $S(h)$ is closer to your conjecture for the same value of h?

c) Explain your results in part (a) when $h = \pm 10^{-15}$.

11. $f(x) = 3x^2 - 2x$ **12.** $f(x) = 2x^3 + 4x$

13. $f(x) = 1/x^2$ **14.** $f(x) = 1/x$

15. $f(x) = |x|$ **16.** $f(x) = |x - 2|$

17. $f(x) = \sqrt{4x - x^2}$ **18.** $f(x) = \sqrt{4 - x^2}$

In Exercises 19–22, explain why you cannot use (a) $S(h)$ or (b) $D(h)$ to approximate the indicated derivative.

19. $f'(0)$ in Exercise 13 **20.** $f'(0)$ in Exercise 14

21. $f'(0)$ in Exercise 15 **22.** $f'(2)$ in Exercise 16

In Exercises 23 and 24, explain why you cannot use $S(h)$ to approximate the indicated derivative.

23. $f'(0)$ in Exercise 17 **24.** $f'(2)$ in Exercise 18

25. Let $f(x) = \sqrt[3]{\dfrac{x^2 \sin(\tan x)}{x^2 \cdot 2^x + x^5 - 4x^2}}$.

a) Use NDER to approximate $f'(-1)$, $f'(0)$, $f'(1.5)$, and $f'(3.5)$.

b) Explain why you must interpret the results of part (a) carefully. (*Hint:* GRAPH $y = f(x)$ in a $[-5, 5]$ by $[-0.5, 0.5]$ window, and consider the domain of f.)

26. Let $g(x) = \dfrac{x^8 - 2x + 5}{(x^2+3)^4 (2 + \sin x)}$.

a) Use NDER to approximate $g'(-2), g'(0), g'(1.5)$, and $g'(5)$.

b) How confident are you about the results of part (a)? Explain. (*Hint:* Investigate the graph of g.)

Consider the functions $y = f(x)$ in Exercises 27–32.

a) GRAPH $y_1 = f(x)$ and $y_2 =$ NDER y_1 in the same viewing window.

b) For what values of x, if any, does $y_1{}'$ fail to exist? Why? How does the graph of $y_2 =$ NDER y_1 help answer this question?

c) For what values of x, if any, is $y_1{}'$ positive? Zero? Negative?

d) For what value of x, if any, is the slope of the line tangent to the curve $y_1 = f(x)$ positive? Zero? Negative?

e) Over what intervals of x-values, if any, does the function $y_1 = f(x)$ increase as x increases? Decrease as x increases? How is this connected with what you found in part (c)?

27. $f(x) = -x^2$ **28.** $f(x) = -\dfrac{1}{x}$

29. $f(x) = \sqrt[3]{x-2}$ **30.** $f(x) = \sqrt[3]{2-x}$

31. $f(x) = \sqrt{1-x}$ **32.** $f(x) = \sqrt[4]{x-1}$

33. GRAPH the first eight terms of the Weierstrass function in the standard viewing window. ZOOM-IN several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

34. GRAPH the first six terms of the Weierstrass function in the standard viewing window. ZOOM-IN several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

For each function f in Exercises 35–37, do the following.

a) Draw the graph of $y =$ NDER $f(x)$ in the $[-3, 3]$ by $[-5, 5]$ viewing window.

b) Describe the graph in part (a). Write an algebraic representation for $y =$ NDER $f(x)$.

c) Compute NDER $(f(x), 0)$. Find $f'(0)$, or explain why it does not exist.

35. $f(x) = \begin{cases} -x^2, & x < 0, \\ 4 - x^2, & x \ge 0. \end{cases}$

36. Let $f(x) = \begin{cases} x^2, & x < 0, \\ x, & x \ge 0. \end{cases}$

37. Let $f(x) = \begin{cases} -x^2/2, & x < 0, \\ x^2/2, & x \ge 0. \end{cases}$

38. Let $f(x) = ax^2 + bx + c, a \ne 0$.

a) Show that $f'(x) = 2ax + b$.

b) Show that the symmetric difference quotient $[f(x+h) - f(x-h)]/(2h) = 2ax + b$ when $h \ne 0$.

c) Explain why NDER $f(x)$ is identical to $f'(x)$.

In Exercises 39–41, GRAPH $y_1 = f(x)$ and $y_2 =$ NDER $f(x)$ in the same viewing window. Find $f'(0)$, or explain why it does not exist.

39. $f(x) = \begin{cases} x^3 + 6x^2 + 12x, & x < 0 \\ -x^2, & x \ge 0 \end{cases}$

40. $f(x) = \begin{cases} -3\sqrt{|x|}, & x < 0 \\ 3 - 0.2x^2, & x \ge 0 \end{cases}$

41. $f(x) = -5\sqrt[3]{|x|}$

42. **a)** Draw the graphs of $y_1 = a^x$ and $y_2 = \text{NDER}(a^x)$ in the $[-1, 2]$ by $[-2, 8]$ viewing window for $a = 0.5$, $a = 0.75$, $a = 1$, $a = 1.5$, $a = 2$ and $a = 3$.
b) Determine a value for a so that the graphs of y_1 and y_2 in part (a) are identical.
c) Find a nontrivial function $y = f(x)$ with the property that $f(x) = f'(x)$ for each value of x in the domain of f. ($f(x) = 0$ is an example of a trivial function for which $f = f'$.)

43. **a)** Draw the graph of $y_1 = \sin x$ and $y_2 = \text{NDER}(\sin x)$ in the $[-10, 10]$ by $[-2, 2]$ viewing window.
b) Make a conjecture about $D_x(\sin x)$.
c) Test your conjecture by graphing your conjecture and $y_2 = \text{NDER}(\sin x)$ in the same viewing window.

44. Let $f(x) = \sqrt[5]{x}$. What are $f'(0)$ and NDER $(f, 0)$? Explain.

45. Let $f(x) = |x|$. What are $f'(0)$ and NDER $(f, 0)$? Explain.

46. Let $f(x) = |x|$ and $g(x) = (f(x+0.01) - f(x-0.01))/0.02$.
a) Show that f is a continuous function.
b) Show that g is a continuous function.
c) Draw the graph of g in the $[-10, 10]$ by $[-2, 2]$ viewing window.
d) Does this graph contradict the continuity of g at $x = 0$? (*Hint:* ZOOM-IN around $x = 0$.)
e) Write g as a piecewise function.

47. Show that the function

$$f(x) = \begin{cases} 0, & -1 \le x < 0, \\ 1, & 0 \le x \le 1, \end{cases}$$

is not the derivative of any function on the interval $-1 \le x \le 1$. (*Hint:* Does f have the intermediate value property on the interval?)

48. Let $f(x) = \sin x$. Use the definition of the derivative to compute $f'(0)$.

49. Show that the greatest integer function $y = [x]$ is not the derivative of any function throughout the interval $-\infty < x < \infty$.

50. Let $S_f(h) = [f(x+h) - f(x-h)]/2h$ be the symmetric difference quotient that is used to compute NDER f. Show that if f is a continuous function, then $S_f(h)$ is a continuous function.

3.3 Differentiation Rules

The process of calculating a derivative is called differentiation. The goal of this section is to show how to differentiate functions rapidly—without having to apply the definition each time. It will then be an easy matter to calculate the velocities, accelerations, and other important rates of change that we will encounter in Section 3.4.

Integer Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero. In short,

> **RULE 1 Derivative of a Constant**
>
> If c is a constant, then
>
> $$\frac{d}{dx}(c) = 0.$$

Proof of Rule 1 If $f(x) = c$ is a function with a constant value c, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$ ▭