Exercises 3.3

In Exercises 1–40, use the rules of differentiation to find the requested derivatives.

In Exercises 1–12, find dy/dx and d^2y/dx^2.

1. $y = x$

2. $y = -x$

3. $y = -x^2 + 3$

4. $y = \frac{x^3}{3} - x$

5. $y = 2x + 1$

6. $y = x^2 + x + 1$

7. $y = \frac{x^3}{3} + \frac{x^2}{2} + x$

8. $y = 1 - x + x^2 - x^3$

9. $y = x^4 - 7x^3 + 2x^2 + 15$

10. $y = 5x^3 - 3x^5$

11. $y = 4x^2 - 8x + 1$

12. $y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + 3$

Find all derivatives of the functions in Exercises 13–16.

13. $y = x^2 - x$

14. $y = \frac{x^3}{3} + \frac{x^2}{2} - 5$

15. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$

16. $y = \frac{x^5}{120}$

In Exercises 17–24, find dy/dx. Find each derivative in two ways: (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

17. $y = (x + 1)(x^2 + 1)$

18. $y = (x + 1)(3 - x^2)$

19. $y = (x - 1)(x^2 + x + 1)$

20. $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

21. $y = (3x - 1)(2x + 5)$

22. $y = (5 - 3x)(4 - 2x)$

23. $y = x^2(x^3 - 1)$

24. $y = x^2\left(x + 5 + \frac{1}{x}\right)$

In Exercises 25–40, find dy/dx. Support your answer by graphing $y_1 = dy/dx$ and $y_2 =$ NDER (y) in the same viewing window.

25. $y = \frac{x - 1}{x + 7}$

26. $y = \frac{2x + 5}{3x - 2}$

27. $y = \frac{x^3 + 7}{x}$

28. $y = \frac{x^2 + 5x - 1}{x^2}$

29. $y = \frac{(x - 1)(x^2 + x + 1)}{x^3}$

30. $y = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

31. $y = (1 - x)(1 + x^2)^{-1}$

32. $y = (2x - 7)^{-1}(x + 5)$

33. $y = \frac{x^2}{1 - x^3}$

34. $y = \frac{x^2 - 1}{x^2 + x - 2}$

35. $y = \frac{10}{\sqrt{x} - 4}$

36. $y = \frac{x}{2\sqrt{x} - 7}$

37. $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

38. $y = \frac{1 + x - 4\sqrt{x}}{x}$

39. $y = \frac{1}{\left(x^2 - 1\right)\left(x^2 + x + 1\right)}$

40. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

In Exercises 41–46, find $y' = dy/dx$ and $y'' = d^2y/dx^2$. Support your answer by graphing $y_1 = dy/dx$ and $y_2 =$ NDER (y) in the same viewing window and by graphing $y_3 = d^2y/dx^2$ and $y_4 =$ NDER2 (y) in the same viewing window.

41. $y = \frac{3}{x^2}$

42. $y = -\frac{1}{x}$

43. $y = \frac{5}{x^4}$

44. $y = -\frac{3}{x^7}$

45. $y = x + 1 + \frac{1}{x}$

46. $y = \frac{12}{x} - \frac{4}{x^3} + \frac{1}{x^4}$

For each function $y = f(x)$ in Exercises 47–50 use NDER to determine the equation of the tangent line at $(a, f(a))$. GRAPH both the function and the tangent line in the same viewing window.

47. $f(x) = x3^{-0.2x}, \quad a = 1$

48. $f(x) = \frac{\sin x}{x}, \quad a = \pi$

49. $f(x) = \frac{x + 3}{x^3 - 2x + 5}, \quad a = 0$

50. $f(x) = \sqrt[3]{\frac{x - 1}{x^2 + 5}}, \quad a = 2$

In Exercises 51–56, determine complete graphs for f' and f'' for each function $y = f(x)$.

51. $y = x \sin x$

52. $y = x^2 \sin x$

53. $y = \frac{2^x}{x^2 - 1}$

54. $y = \frac{3^x}{4 - x^2}$

55. $y = \sqrt[3]{\frac{x + 3}{x - 5}}$

56. $y = \sqrt[3]{\frac{x + 1}{x^2 + 2}}$

57–62. For each function in Exercises 51–56, solve $f'(x) = 0$ and $f''(x) > 0$.

63. Let f be the function $f(x) = (2x - 5)/(3x^2 + 4)$. Determine $f''(x)$. Try to solve $f''(x) > 0$ *exactly*. Solve $f''(x) > 0$ using any appropriate method.

64. Use the definition of derivative (given in Section 3.1, Eq. (1)) to show that

a) $\frac{d}{dx}(x) = 1.$

b) $\frac{d}{dx}(-u) = -\frac{du}{dx}.$

65. Use the product rule to show that $\frac{d}{dx}c \cdot f(x) = c \cdot \frac{d}{dx}f(x)$ for any constant c.

66. Devise a rule for $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right)$.

67. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$

c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

68. Suppose u and v are functions of x that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$

c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(3u - 2v + 2uv)$

69. Which of the following numbers is the slope of the line tangent to the curve $y = x^2 + 5x$ at $x = 3$?

a) 24 **b)** $-5/2$ **c)** 11 **d)** 8

70. Which of the following numbers is the slope of the line $3x - 2y + 12 = 0$?

a) 6 **b)** 3 **c)** $3/2$ **d)** $2/3$

In Exercises 71–76, support your answers graphically.

71. Find the equation of the line perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

72. Find the tangents to the curve $y = x^3 + x$ at the points where the slope is 4. What is the smallest slope on the curve? At what value of x does the curve have this slope?

73. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

74. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

75. Find the tangents to *Newton's Serpentine*,

$$y = \frac{4x}{x^2 + 1}$$

at the origin and the point $(1, 2)$.

76. Find the tangent to the *Witch of Agnesi*

$$y = \frac{8}{4 + x^2}$$

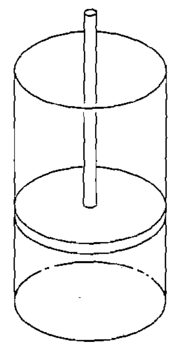
at the point $(2, 1)$. (There is a nice story about the name of this curve in the historical note on Maria Agnesi in Chapter 10.)

When we work with functions of a single variable in mathematics, we often call the independent variable x and the dependent variable y. Applied fields use many different letters, however. Here are some examples. In these cases, the exact derivative is very useful.

77. *Cylinder pressure.* If the gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

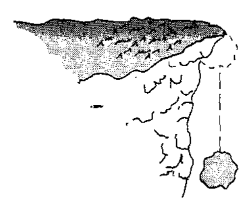
in which a, b, n, and R are constants. Find dP/dV.



78. *Free fall.* When a rock falls from rest near the surface of the earth, the distance it covers during the first few seconds is given by the equation

$$s = 4.9t^2.$$

In this equation, s is the distance in meters, and t is the elapsed time in seconds. Find ds/dt and d^2s/dt^2.



79. *The body's reaction to medicine.* The reaction of the body to a dose of medicine can often be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine. In Chapter 4, we shall see how to find the amount of medicine to which the body is most sensitive. (*Source: Some Mathematical Models in Biology*, Revised Edition, December 1967, PB-202 364, p. 221; distributed by N.T.I.S., U.S. Department of Commerce.)

80. Let K, m, n, a, and b be positive constants. Explain why horizontal tangent lines to the graphs of $y = Ka^x/(m + nb^x)$ and $y = a^x/(m + nb^x)$ have the same x-coordinate. Determine the x-coordinates of all points where the tangent lines are horizontal for $a = 2, m = 2, n = 1$, and $b = 3$. Draw complete graphs for $K = 1, K = 2$, and $K = 3$ in the same viewing window. Describe a technique, suggested by this exercise, that you could employ using a graphing calculator to help you investigate a function.

81. Show that if f is an even function, then f' is an odd function.

82. Show that if f is an odd function, then f' is an even function.

Exercises 83 and 84 refer to the function $f(x) = y_1 = x2^{-x}$ of Example 9 and Fig. 3.23.

83. **a)** Find the x-intercept of $y_2 =$ NDER y_1 shown in Fig. 3.23.
b) Find the coordinates of the local maximum of f shown in Fig. 3.23.
c) Compare the x-coordinate of the point in part (b) with the x-intercept in part (a).

84. **a)** Find the x-intercept of $y_3 =$ NDER $2f$ shown in Fig. 3.23.
b) Find the coordinates of the local minimum of $y_2 =$ NDER y_1 shown in Fig. 3.23.
c) Compare the x-coordinate of the point in part (b) with the x-intercept in part (a).

85. *Third derivatives.* Most graphing calculators that have a numerical derivative procedure (NDER) allow only one "nested" computation. This permits quick numerical computation of second derivatives as NDER2f = NDER (NDER f). However, NDER (NDER (NDERf)) is *not* allowed. Let $y_1 =$ NDER (NDER f) for $f(x) = x^5 - 3x^4 + x^3 - 6x^2 + 7x - 5$.
a) Compute the maximum error in using y_1 as an estimate for $y = f''(x)$ for $-10 \leq x \leq 10$. Explain how you arrived at your solution.
b) Use the symmetric difference quotient to determine a function y_2 that closely approximates $y = f^{(3)}(x)$.
c) Compute the maximum error in using y_2 as an estimate for $y = f^{(3)}(x)$ for $-10 \leq x \leq 10$. Explain how you arrived at your solution.

3.4 Velocity, Speed, and Other Rates of Change

In this section, we see how derivatives provide the mathematics we need to understand the way in which things change in the world around us. With derivatives, we can describe the rates at which water reservoirs empty, populations change, rocks fall, the economy evolves, and an athlete's blood sugar varies with exercise. We begin with free fall, the kind of fall that takes place in a vacuum near the surface of Earth.

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time elapsed. At least, that is what happens when the body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of their fall through air, before their velocities build up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way in which the body falls *free fall*.