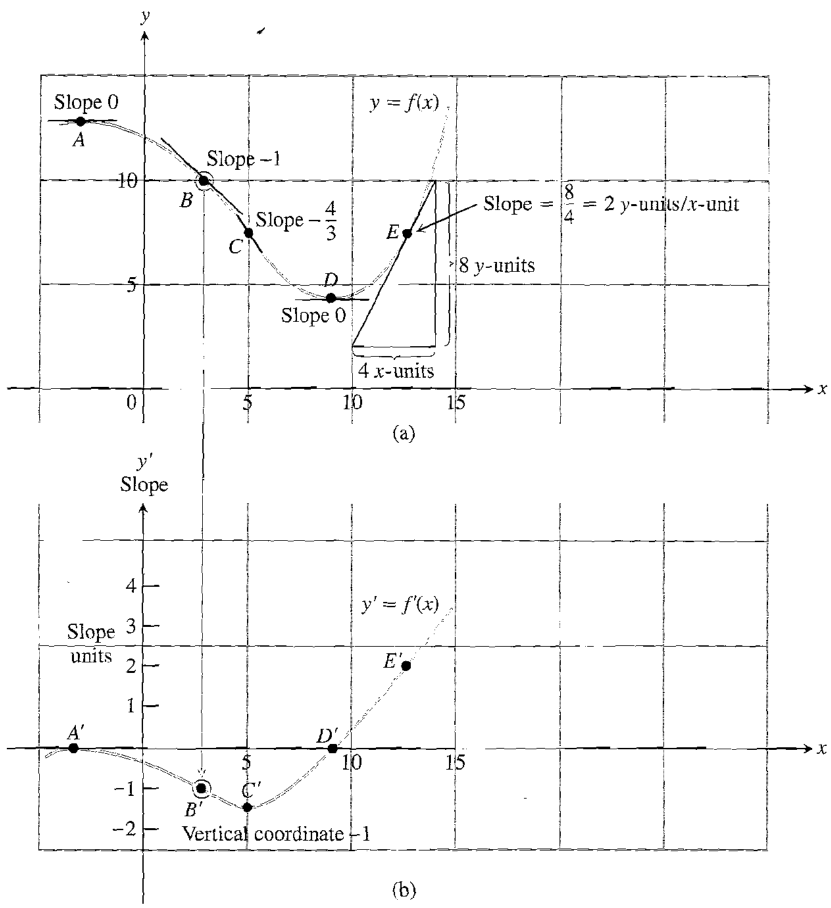
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.



3.30 We made the graph of $y' = f'(x)$ in part (b) by plotting slopes from the graph of $y = f(x)$ in part (a). The vertical coordinate, -1, of B' is the slope at B, and so on. The graph of $y' = f'(x)$ is a visual record of how the slope of f changes with x.

Exercises 3.4

The equations in Exercises 1–6 give the position $s = f(t)$ of a particle moving along the line $y = 3$; s is in meters and t is in seconds.

a) Use parametric mode to simulate the motion of the particle for the indicated values of t. Describe the path of the particle.

b) Determine the position of the particle at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ seconds.

c) Determine where and at what time the direction of the particle changes (if it does). Also determine the velocity and acceleration at these times.

d) Determine the total distance the particle travels for the specified t interval.

e) Simultaneously GRAPH the path of the particle and its position as a function of time.

f) Simultaneously GRAPH the path of the particle, the velocity function $y = v(t) = s'(t)$, and the acceleration function

$y = a(t) = s''(t)$. When is the particle at rest?

1. $s = t^2 - 3t + 2, \quad 0 \le t \le 5$
2. $s = 5 + 3t - t^2, \quad 0 \le t \le 5$
3. $s = t^3 - 6t^2 + 7t - 3, \quad 0 \le t \le 5$
4. $s = 4 - 7t + 6t^2 - t^3, \quad 0 \le t \le 5$
5. $s = t \sin t, \quad 0 \le t \le 15$
6. $s = 5 \sin\left(\frac{2t}{\pi}\right), \quad 0 \le t \le 2\pi$

7. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are: Mars, $s = 1.86t^2$; Jupiter, $s = 11.44t^2$. How long would it take a rock falling from rest to reach a velocity of 16.6 m/sec (about 60 km/h) on each planet?

8. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.
   a) Find the rock's velocity and acceleration as a function of time. (The acceleration in this case is the acceleration of gravity on the moon.)
   b) How long did it take the rock to reach its highest point?
   c) How high did the rock go?
   d) How long did it take the rock to reach half its maximum height?
   e) How long was the rock aloft?

9. Devise a grapher simulation of the problem situation in Exercise 8. Use it to support the answers obtained analytically.

10. On Earth, in the absence of air, the rock in Exercise 8 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds. How high would the rock go?

11. A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t seconds. How long would it take the bullet to get back down in each case?

12. Devise a grapher simulation of the problem situation in Exercise 11. Use it to support the answers obtained analytically.

13. When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while but then stopped growing and began to decline. The size of the population at time t (hours) was $b(t) = 10^6 + 10^4 t - 10^3 t^2$. Find the growth rates at $t = 0$, $t = 5$, and $t = 10$ hours.

14. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

15. *Marginal cost.* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.
    a) Find the average cost of producing 100 washing machines.
    b) Find the marginal cost when 100 washing machines are produced.
    c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

16. *Marginal revenue.* Suppose the weekly revenue in dollars from selling x custom-made office desks is
$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$
    a) Draw a complete graph of r. What values of x make sense in this problem situation?
    b) Find the marginal revenue when x desks are sold.
    c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.
    d) Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

17. The position of a body at time t sec is $s = t^3 - 6t^2 + 9t$ m. Find the body's acceleration each time the velocity is zero.

18. A body's velocity at time t sec is $v = 2t^3 - 9t^2 + 12t - 5$ m/sec. Find the body's speed each time the acceleration is zero.

19. Determine complete graphs of the following parametric equations.
    a) $x(t) = 3t - 5\sin t, \quad y(t) = 3t - 5\cos t$
    b) $x(t) = \dfrac{6\cos t}{4 - 3\cos t}, \quad y(t) = \dfrac{6\sin t}{4 - 3\cos t}$
    c) $x(t) = 2 - 7\cos t, \quad y(t) = -2 + 3\cos t$

20. Which of the graphs in Exercise 19 are graphs of functions?

21. Let $f'(x) = 3x^2$.
    a) Compute the derivatives of $g(x) = x^3$, $h(x) = x^3 - 2$, and $t(x) = x^3 + 3$.
    b) Graph the numerical derivatives of g, h, and t.
    c) Describe the *family* of functions, $f(x)$, that have the property that $f'(x) = 3x^2$.
    d) Is there a unique f such that $f'(x) = 3x^2$ *and* $f(0) = 0$? What is it?
    e) Is there a unique g such that $g'(x) = 3x^2$ *and* $g(0) = 3$? What is it?

22. The monthly profit (in thousands of dollars) of a software company is given by
$$P(x) = \frac{10}{1 + 50 \cdot 2^{5-0.1x}},$$
where x is the number of software packages sold.
    a) Draw a complete graph of $y = P(x)$.
    b) What values of x make sense in the problem situation?

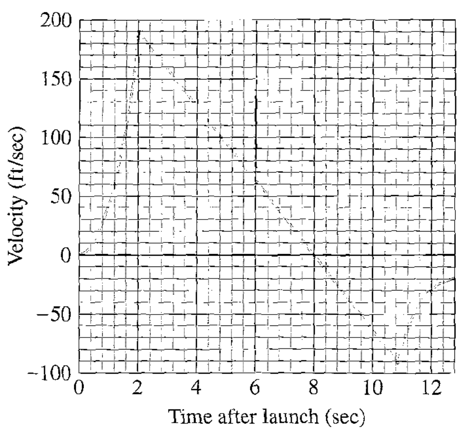
c) Draw a graph of $y = P'(x)$ (Use $y =$ NDER $P(x)$.) Compare with the graph of $y = M(x)$ in Example 14.

d) What is the profit when the marginal profit is maximum? What is the marginal profit when 50 units are produced? 100 units, 125 units, 150 units, 175 units, and 300 units?

e) What is $\lim_{x\to\infty} P(x)$? What is the maximum profit possible?

f) Is there a practical explanation to the maximum profit question? Explain your reasoning.

23. When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts downward. The parachute slows the rocket to keep it from breaking when it lands. This graph shows velocity data from the flight.



Use the data to answer the following.

a) How fast was the rocket climbing when the engine stopped?

b) For how many seconds did the engine burn?

c) When did the rocket reach its highest point? What was its velocity then?

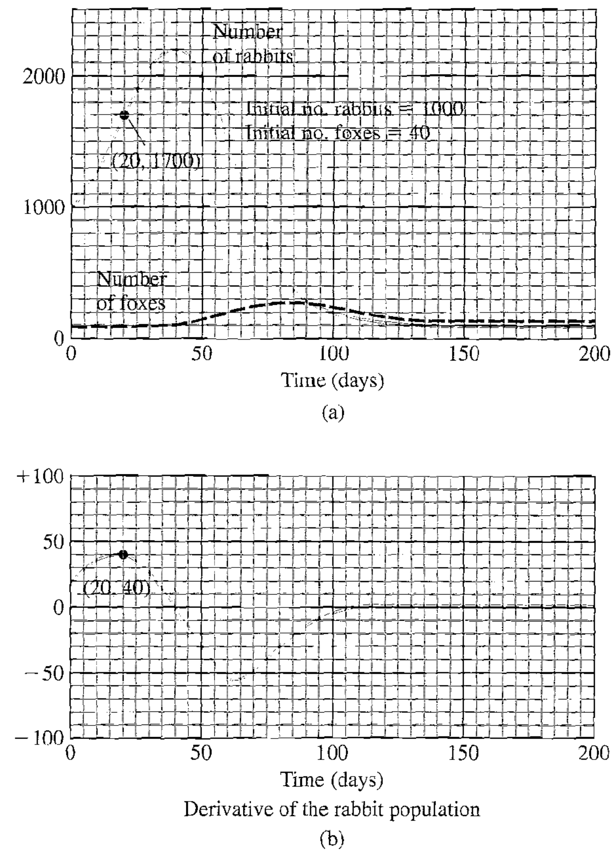
d) When did the parachute pop out? How fast was the rocket falling then?

e) How long did the rocket fall before the parachute opened?

f) When was the rocket's acceleration greatest? When was the acceleration constant?

24. *Pisa by parachute (continuation of Exercise 23).* A few years ago, Mike McCarthy parachuted 179 ft from the top of the Tower of Pisa. Make a rough sketch to show the shape of the graph of his downward velocity during the jump.

Exercises 25 and 26 are about the graphs in Fig. 3.39. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits, and as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.39(b) shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 16.



3.39 Rabbits and foxes in an arctic predator–prey food chain. (*Source: Differentiation,* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, Mass. (1975), p. 86.)

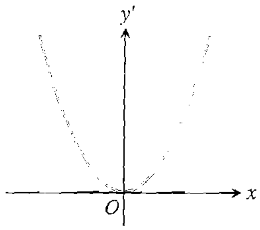
25. a) What is the value of the derivative of the rabbit population when the number of rabbits is largest? Smallest?

b) What is the size of the rabbit population when its derivative is largest? Smallest?

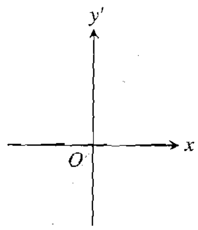
c) In what units should the slopes of the rabbit and fox population curves be measured?

26. Clearly there cannot be a fractional number of rabbits. Explain then, why the graphs appear to be continuous, and also how they should be interpreted.

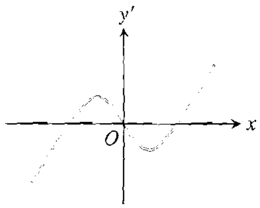
Match the graphs of the functions in Exercises 27–30 with the graphs of the derivatives shown here:



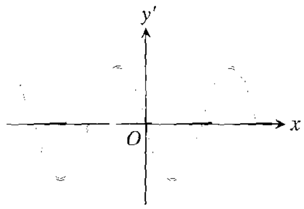
(a)



(b)

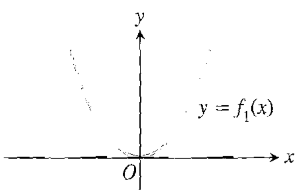


(c)

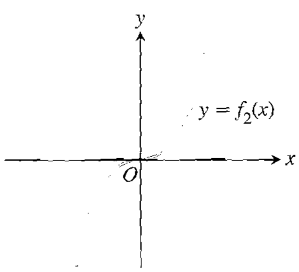


(d)

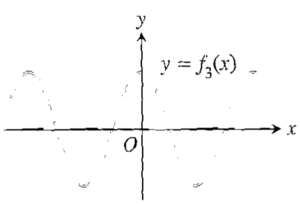
27.



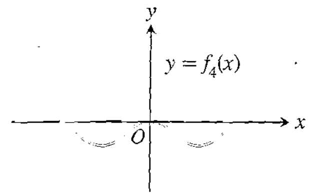
28.



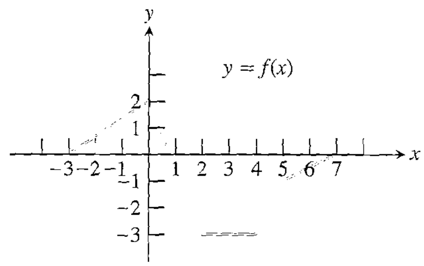
29.



30.



31. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



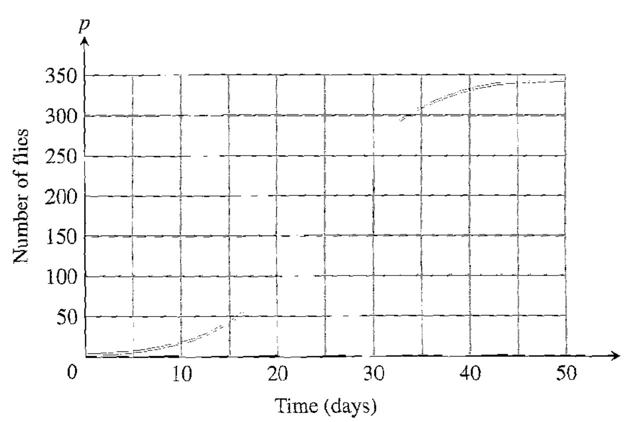
a) Graph the function's derivative.

b) At what values of x between $x = -3$ and $x = 7$ is the derivative not defined?

32. *Fruit flies (Example 1, Section 3.1 continued).* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

a) Use the graphical technique of Example 16 to graph the derivative of the fruit fly population introduced in Section 3.1. The graph of the population is reproduced below. What units should be used on the horizontal and vertical axes for the derivative's graph?

b) During what days does the population seem to be increasing fastest? Slowest?



33. In Exploration 2, at what time is the particle at the point (5, 2)?

34. The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $s(t) = 2t^3 - 13t^2 + 22t - 5$ where t is time in seconds.

a) Describe the motion of the particle for $t \geq 0$.
b) When does the particle speed up? Slow down?
c) When does the particle change direction?
d) When is the particle at rest?
e) Describe the velocity and speed of the particle.
f) When is the particle at the point (5, 2)?

The data in Exercises 35 and 36 give the coordinates s of a moving body for various values of t. Plot s versus t on coordinate paper, and sketch a smooth curve through the given points. Assuming that this smooth curve represents the motion of the body, estimate the velocity at (a) $t = 1.0$; (b) $t = 2.5$, (c) $t = 3.5$.

35.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	12.5	26	36.5	44	48.5	50	48.5	44	36.5

36.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	3.5	−4	−8.5	−10	−8.5	−4	3.5	14	27.5

3.5 Derivatives of Trigonometric Functions

As we mentioned in Section 1.7, trigonometric functions are important because so many of the phenomena about which we want information are periodic (heart rhythms, earthquakes, tides, weather). Continuous periodic functions can always be expressed in terms of sines and cosines, so the derivatives of sines and cosines play a key role in describing and predicting important changes. This section shows how to differentiate the six basic trigonometric functions.

For $y_1 = \sin x$, graphs of $y_2 = \text{NDER}\, y_1$ and $y_3 = \cos x$ in the same viewing window of a grapher strongly suggest that the derivative of the sine function is the cosine function. We now confirm this analytically.

EXPLORATION BIT

Let

$$y_1 = \sin x,$$

$$y_2 = \text{NDER}\, y_1,$$

$$y_3 = \cos x.$$

GRAPH y_2 and y_3. Make a conjecture.

A New Limit

First, we have

$$\begin{aligned}\frac{\cos h - 1}{h} &= -\frac{1 - \cos h}{h}\\ &= -\frac{1 - \cos 2(h/2)}{2(h/2)}\\ &= -\frac{\sin^2(h/2)}{h/2} \qquad \text{Section 1.7, half-angle formula}\\ &= -\sin(h/2)\cdot\frac{\sin(h/2)}{h/2}.\end{aligned}$$

Then,

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0}\left(-\sin(h/2)\cdot\frac{\sin(h/2)}{h/2}\right)\\ &= \lim_{h\to 0}(-\sin(h/2))\cdot\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\\ &= 0\cdot 1 = 0.\end{aligned}$$