EXPLORATION 3
Support Graphically

Use rules of differentiation to find y' for each function $y = f(x)$. Support graphically by comparing the graph of your result with the graph of NDER $f(x)$.

1. $y = 3x + \cot x$
2. $y = 2/\sin x$

Find y'' for each function $y = f(x)$. Support graphically by comparing the graph of your result with the graph of NDER2 $f(x)$.

3. $y = \sec x$
4. $y = 2\sin x + 3\cos x$

Exercises 3.5

In Exercises 1–24, find dy/dx. Support your answer by graphing the numerical derivative and comparing with the graph of the exact derivative.

1. $y = 1 + x - \cos x$
2. $y = 2\sin x - \tan x$
3. $y = \frac{1}{x} + 5\sin x$
4. $y = x^2 - \sec x$
5. $y = \csc x - 5x + 7$
6. $y = 2x + \cot x$
7. $y = x\sec x$
8. $y = x\csc x$
9. $y = x^2\cot x$
10. $y = 4 - x^2\sin x$
11. $y = 3x + x\tan x$
12. $y = x\sin x + \cos x$
13. $y = \sin x\sec x$
14. $y = \sec x\csc x$
15. $y = \tan x\cot x$
16. $y = \cos x(1 + \sec x)$
17. $y = \frac{4}{\cos x}$
18. $y = 5 + \frac{1}{\tan x}$
19. $y = \frac{\cos x}{x}$
20. $y = \frac{2}{\csc x} - \frac{1}{\sec x}$
21. $y = \frac{x}{1 + \cos x}$
22. $y = \frac{\sin x + \cos x}{\cos x}$
23. $y = \frac{\cot x}{1 + \cot x}$
24. $y = \frac{\cos x}{1 + \sin x}$
25. Find y'' if $y = \csc x$.
26. Find $y^{(4)} = d^4y/dx^4$ if
    a) $y = \sin x$,
    b) $y = \cos x$.

In Exercises 27–30, find equations for the lines that are tangent and normal to the curve $y = f(x)$ at the given point $(x, f(x))$. Support your answers graphically using square viewing windows.

27. $y = \sin x, \quad x = 0$
28. $y = \tan x, \quad x = 0$
29. $y = 2\sin^2 x, \quad x = 2$
30. $y = \frac{2 + \cot x}{x}, \quad x = 1$
31. Prove that $D_x \cos x = -\sin x$.
32. Show that the graphs of $y = \sec x$ and $y = \cos x$ have horizontal tangents at $x = 0$.
33. Show that the graphs of $y = \tan x$ and $y = \cot x$ never have horizontal tangents.

Do the graphs of the functions in Exercises 34–37 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not?

34. $y = x + \sin x$
35. $y = 2x + \sin x$
36. $y = x + \cos x$
37. $y = x + 2\cos x$
38. *Simple harmonic motion.* The equations in parts (a) and (b) give the position $s = f(t)$ of a body moving along a coordinate line. Find each body's velocity, speed, and acceleration at time $t = \pi/4$.
    a) $s = 2 - 2\sin t$
    b) $s = \sin t + \cos t$
39. Find equations for the lines that are tangent and normal to the curve $y = \sqrt{2}\cos x$ at the point $(\pi/4, 1)$.

40. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$.

Find equations for the horizontal tangents to the graphs in Exercises 41 and 42.

41. $y = \cot x - \sqrt{2}\csc x$, $0 < x < \pi$

42. $y = \tan x + 3\cot x - 3$, $0 < x < \pi/2$

43. GRAPH $y = \tan x$ and its derivative together over the interval $-\pi/2 < x < \pi/2$.

44. GRAPH $y = \cot x$ and its derivative together for $0 < x < \pi$.

45. Although $\lim_{h\to 0}(1 - \cos h)/h = 0$, it turns out that

$$\lim_{h\to 0}\frac{1-\cos h}{h^2} \neq 0.$$

Determine the limit. Confirm analytically.

46. Derive Eq. (2) by writing $\sec x = 1/\cos x$ and differentiating with respect to x.

47. Derive Eq. (3) by writing $\cot x = (\cos x)/(\sin x)$ and differentiating with respect to x.

48. Derive Eq. (4) by writing $\csc x = 1/\sin x$ and differentiating with respect to x.

3.6 The Chain Rule

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is: with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriately related points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it.

Introductory Examples

Some examples will help to show what is going on.

EXAMPLE 1

The function $y = 9x^2 + 6x + 1 = (3x + 1)^2$ is the composite of $y = u^2$ and $u = 3x + 1$. How are the derivatives of these three functions related?

$$\frac{dy}{dx} = \frac{d}{dx}(9x^2 + 6x + 1) = 18x + 6 = 6(3x + 1) = 6u$$
$$\frac{dy}{du} = \frac{d}{du}(u^2) = 2u$$
$$\frac{du}{dx} = \frac{d}{dx}(3x + 1) = 3.$$

Because $6u = 2u \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$