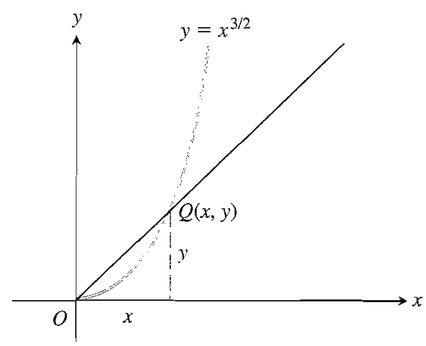
EXAMPLE 7

Find the derivative of $y = x^{3/2}$ at $x = 0$.

Solution When the graph of a function stops abruptly at a point, as the graph of $y = x^{3/2}$ does at $x = 0$ (Fig. 3.51), we calculate its derivative as a one-sided limit. The Power Rule still applies, giving in this case

$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{3}{2}x^{1/2}\right|_{x=0} = 0.$$



3.51 The graph of $y = x^{3/2}$. The slope of the curve at $x = 0$ is $\lim_{Q \to O} m_{OQ} = \lim_{x \to 0^+} \left(\frac{y}{x}\right) = 0.$

We can see why this equation holds by looking at the geometry of the curve. The slope of a typical secant line through the origin and a point $Q(x, y)$ on the curve is

$$m_{OQ} = \frac{y - 0}{x - 0} = \frac{x^{3/2}}{x} = x^{1/2}.$$

As Q approaches the origin from the right, m_{OQ} approaches zero, in agreement with the result from the Power Rule.

Exercises 3.7

Find dy/dx in Exercises 1–26.

1. $y = x^{9/4}$
2. $y = x^{-3/5}$
3. $y = \sqrt[3]{x}$
4. $y = \sqrt[4]{x}$
5. $y = (2x + 5)^{-1/2}$
6. $y = (1 - 6x)^{2/3}$
7. $y = x\sqrt{x^2 + 1}$
8. $y = \dfrac{x}{\sqrt{x^2 + 1}}$
9. $x^2 y + xy^2 = 6$
10. $x^3 + y^3 = 18xy$
11. $2xy + y^2 = x + y$
12. $x^3 - xy + y^3 = 1$
13. $x^2 y^2 = x^2 + y^2$
14. $(3x + 7)^2 = 2y^3$
15. $y^2 = \dfrac{x - 1}{x + 1}$
16. $x^2 = \dfrac{x - y}{x + y}$
17. $y = \sqrt{1 - \sqrt{x}}$
18. $y = 3(2x^{-1/2} + 1)^{-1/3}$
19. $y = 3(\csc x)^{3/2}$
20. $y = [\sin(x + 5)]^{5/4}$
21. $x = \tan y$
22. $x = \sin y$
23. $x + \tan(xy) = 0$
24. $x + \sin y = xy$
25. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
26. $y^2 \cos\left(\dfrac{1}{y}\right) = 2x + 2y$
27. Which of the following could be true if $f''(x) = x^{-1/3}$?

    a) $f(x) = \dfrac{3}{2}x^{2/3} - 3$
    b) $f(x) = \dfrac{9}{10}x^{5/3} - 7$
    c) $f'''(x) = -\dfrac{1}{3}x^{-4/3}$
    d) $f'(x) = \dfrac{3}{2}x^{2/3} + 6$

28. Which of the following could be true if $g''(t) = 1/t^{3/4}$?

a) $g'(t) = 4\sqrt[4]{t} - 4$ b) $g'''(t) = -4/\sqrt[4]{t}$

c) $g(t) = t - 7 + (16/5)t^{5/4}$

d) $g'(t) = (1/4)t^{1/4}$

In Exercises 29–34, use implicit differentiation to find dy/dx and then d^2y/dx^2.

29. $x^2 + y^2 = 1$ **30.** $x^{2/3} + y^{2/3} = 1$

31. $y^2 = x^2 + 2x$ **32.** $y^2 + 2y = 2x + 1$

33. $y + 2\sqrt{y} = x$ **34.** $xy + y^2 = 1$

In Exercises 35–44, find the lines that are (a) tangent and (b) normal to the curve at the given point.

35. $x^2 + xy - y^2 = 1, \quad (2, 3)$

36. $x^2 + y^2 = 25, \quad (3, -4)$

37. $x^2y^2 = 9, \quad (-1, 3)$

38. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$

39. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$

40. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$

41. $2xy + \pi \sin y = 2\pi, \quad (1, \pi/2)$

42. $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$

43. $y = 2\sin(\pi x - y), \quad (1, 0)$

44. $x^2 \cos^2 y - \sin y = 0, \quad (0, \pi)$

45. Assume that the equation $2xy + \pi \sin y = 2\pi$ defines y as a differentiable function of x. Evaluate dy/dx when $x = 1$ and $y = \pi/2$.

46. Find an equation for the tangent to the curve $x \sin 2y = y \cos 2x$ at the point $(\pi/4, \pi/2)$.

47. *The eight curve.*

a) Find the slopes of the figure-eight–shaped curve

$$y^4 = y^2 - x^2$$

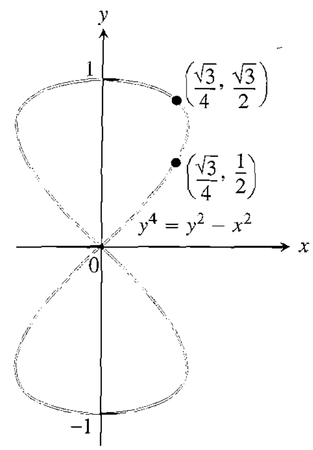
at the two points shown on the graph that follows.

b) Use parametric mode and the two pairs of parametric equations

$$x_1(t) = \sqrt{y^2 - y^4}, \quad y_1(t) = t,$$

$$x_2(t) = -\sqrt{y^2 - y^4}, \quad y_2(t) = t,$$

to show the curve in a viewing window.



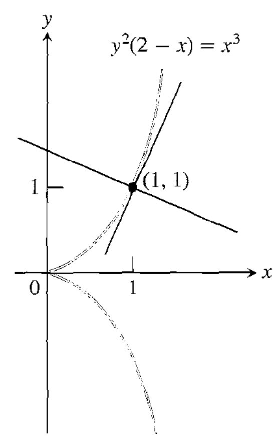
48. *The Cissoid of Diocles (dates from about 200 B.C.).*

a) Find equations for the tangent and normal to the Cissoid of Diocles,

$$y^2(2 - x) = x^3,$$

at the point $(1, 1)$ as pictured below.

b) Use parametric mode to reproduce the curve and the tangent and normal lines at $(1, 1)$.



49. a) Confirm that $(-1, 1)$ is on the curve defined by $x^3y^2 = \cos(\pi y)$.

b) Use part (a) to find the slope of the line tangent to the curve at $(-1, 1)$.

50. a) Show that the relation

$$y^3 - xy = -1$$

cannot be a function of x by showing that there is more than one possible y-value when $x = 2$.

b) On a small enough square with center $(2, 1)$, the part of the graph of the relation within the square will define a function $y = f(x)$. For this function, find $f'(2)$ and $f''(2)$.

51. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

52. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. (In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?)

53. *Orthogonal curves.* Two curves are *orthogonal* at a point of intersection if their tangents there cross at right angles. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal at (1, 1) and (1, −1). Use parametric mode to draw the curves and to show the tangent lines.

54. The position of a body moving along a coordinate line at time t is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the body's velocity and acceleration when $t = 6$ sec.

55. The velocity of a falling body is $v = k\sqrt{s}$ meters per second (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

56. Use parametric mode to GRAPH the curve given by the equation in Example 2. Show that it is a function. Find its domain. GRAPH its derivative.

57. Consider the equation $y^5 + \sin xy = 0$.

a) Show that $-1 \le y \le 1$.

b) Show that $xy = \sin^{-1}(-y^5) + 2k\pi$ or $xy = \pi - \sin^{-1}(-y^5) + 2k\pi$, k any integer.

c) GRAPH the first relation in part (b) parametrically for $k = 0$ by setting

$$x(t) = (1/t)\sin^{-1}(-t^5), \quad y(t) = t.$$

What are the domain and range of this relation?

d) GRAPH the relation in part (b) parametrically for $k = 1$. What are the domain and range of this relation?

58. Consider the relation $x = (\sin^{-1}(-y^5))/y$ graphed in Exercise 57.

a) Find x when $y = -1/2$.

b) Find dy/dx using analytic methods and compute its value at $y = -1/2$.

c) Draw the tangent line to the graph in Exercise 57(c) at $y = -1/2$.

59. Refer to Example 3. There are two points at which dy/dx does not exist. Find them.

60. Draw complete graphs of $y = x^{1/3}$ and its derivative. Find the domain and range of each function.

3.8 Linear Approximations and Differentials

Contrary to what you may think, we have not been using *dx* as a symbol already. We have been using "*dx*" only as *part* of the derivative symbol *d/dx*.

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications. It is important to know how to do this, and in this section we study the simplest of the useful approximations. For reasons that will be clear in a moment, the approximation is called a *linearization*.

We also introduce a new symbol, dx, for an increment in a variable x. This symbol is called the *differential* of x. In the physical sciences, it is used more frequently than Δx. In mathematics, differentials are used to estimate changes in function values, as we shall see toward the end of this section.

EXPLORATION 1

Seeing a Local Approximation

Set your ZOOM factor at 4. In the standard viewing window, GRAPH $f(x) = x^2 - x - 3$ and the line tangent at $(2, f(2))$, namely, $y - f(2) = f'(2)(x - 2)$. ZOOM-IN at the point $(2, f(2))$. ZOOM-IN a second time, then a third time. Explain what you see.