> If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by an equation of the form
>
> $$\Delta y = f'(a)\Delta x + \epsilon \Delta x, \tag{4}$$
>
> in which $\epsilon \to 0$ as $\Delta x \to 0$.

Surprising as it may seem, just knowing the form of Eq. (4) enables us to bring the proof of the Chain Rule to a successful conclusion. You can find out what we mean by turning to Appendix 3.

Formulas for Differentials

TABLE 3.2 Formulas for Differentials

$$d(c) = 0$$
$$d(cu) = c\,du$$
$$d(u+v) = du + dv$$
$$d(uv) = u\,dv + v\,du$$
$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$
$$d(u^n) = nu^{n-1}du$$
$$d(\sin u) = \cos u\,du$$
$$d(\cos u) = -\sin u\,du$$
$$d(\tan u) = \sec^2 u\,du$$
$$d(\cot u) = -\csc^2 u\,du$$
$$d(\sec u) = \sec u \tan u\,du$$
$$d(\csc u) = -\csc u \cot u\,du$$

Every formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

has a corresponding differential formula like

$$d(u+v) = du + dv$$

that comes from multiplying both sides by dx.

To find dy when y is a differentiable function of x, we may either find dy/dx and multiply by dx or use one of the formulas in Table 3.2.

EXAMPLE 10

a) $d(3x^2 - 6) = 6x\,dx$

b) $d(\cos 3x) = -(\sin 3x)d(3x) = -3\sin 3x\,dx$

c) $$d\frac{x}{(x+1)} = \frac{(x+1)dx - x\,d(x+1)}{(x+1)^2} = \frac{x\,dx + dx - x\,dx}{(x+1)^2} = \frac{dx}{(x+1)^2}$$

Notice that a differential on one side of an equation always calls for a differential on the other side of the equation. Thus, we never have $dy = 3x^2$ but, instead, $dy = 3x^2dx$.

Exercises 3.8

In Exercises 1–6, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^4$ at $x = 1$

2. $f(x) = x^{-1}$ at $x = 2$

3. $f(x) = x^3 - x$ at $x = 1$

4. $f(x) = x^3 - 2x + 3$ at $x = 2$

5. $f(x) = \sqrt{x}$ at $x = 4$

6. $f(x) = \sqrt{x^2 + 9}$ at $x = -4$

You want linearizations that will replace the functions in Exercises 7–12 over intervals that include the given points x_0. To make your subsequent work as simple as possible, you want to center each linearization not at x_0 but at a nearby integer $x = a$ at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

7. $f(x) = x^2 + 2x, \quad x_0 = 0.1$

8. $f(x) = x^{-1}, \quad x_0 = 0.6$

9. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -0.9$

10. $f(x) = 1 + x, \quad x_0 = 8.1$

11. $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$

12. $f(x) = \dfrac{x}{x+1}, \quad x_0 = 1.3$

In Exercises 13–18, find the linearization $L(x)$ of the given function at $x = a$. Then graph f and L together near $x = a$.

13. $f(x) = \sin x$ at $x = 0$

14. $f(x) = \cos x$ at $x = 0$

15. $f(x) = \sin x$ at $x = \pi$

16. $f(x) = \cos x$ at $x = -\pi/2$

17. $f(x) = \tan x$ at $x = \pi/4$

18. $f(x) = \sec x$ at $x = \pi/4$

19. Use the linearization $(1+x)^k \approx 1 + kx$ to find linear approximations of the following functions for values of x near zero. Graph each function and its linearization in the $[-2, 2]$ by $[-2, 2]$ viewing window.

a) $(1+x)^2$ **b)** $\dfrac{1}{(1+x)^5}$

c) $\dfrac{2}{1-x}$ **d)** $(1-x)^6$

e) $3(1+x)^{1/3}$ **f)** $\dfrac{1}{\sqrt{1+x}}$

20. Use the linearization $(1+x)^k \approx 1 + kx$ to estimate and compare with a calculator value.

a) $(1.002)^{100}$ **b)** $\sqrt[3]{1.009}$

21. Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x+1}$ and $\sin x$?

22. We know from the Power Rule that the equation

$$\frac{d}{dx}(1+x)^k = k(1+x)^{k-1}$$

holds for every rational number k. In Section 7.3, we shall show that it holds for every irrational number as well. Assuming this result for now, verify Eq. (2) by showing that the linearization of $f(x) = (1+x)^k$ at $x = 0$ is $L(x) = 1 + kx$ for any number k.

23. a) Compute $\sqrt{2}$, $\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{\sqrt{2}}}$ and so forth. At each step, show that the decimal part of the square root is about half the decimal part of its radicand. Informally, we might write: $\sqrt{1.x} \approx 1.(x/2)$.

b) Write a paragraph explaining what is happening in part (a) using the linearization of $\sqrt{1+x}$ at $x = 0$. (See Example 1.)

c) Make a conjecture about what the numbers in part (a) are approaching if the process of taking square roots is continued. Use the linearization of $\sqrt{1+x}$ at $x = 0$ to give a convincing argument for your conjecture.

d) Repeat parts (a)–(c), replacing 2 by other numbers greater than 1.

24. *Continuation of Exercise 23.* One way to describe the halving of the decimal parts of the square roots in Exercise 23 is to say that the number-line distance between 1 and the square root is approximately halved each time.

a) What happens if you start with a positive number that is less than 1 instead of greater than 1? Try it with 0.5.

b) Do the successive square roots approach a number? Can you use the linearization of $\sqrt{1+x}$ at x to explain?

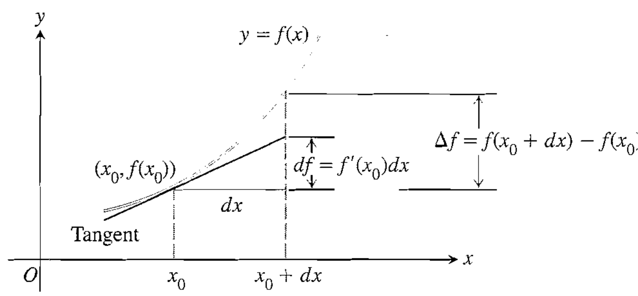
c) Extend the ideas in this and the previous exercise to include $\sqrt[10]{2}$ and $\sqrt[10]{0.5}$.

In Exercises 25–30, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a) the change $\Delta f = f(x_0 + dx) - f(x_0)$;

b) the value of the estimate $df = f'(x_0)dx$; and

c) the error $|\Delta f - df|$.



25. $f(x) = x^2 + 2x, \quad x_0 = 0, \quad dx = 0.1$

26. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

27. $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$

28. $f(x) = x^4, \quad x_0 = 1, \quad dx = 0.1$

29. $f(x) = x^{-1}, \quad x_0 = 0.5, \quad dx = 0.1$

30. $f(x) = x^3 - 2x + 3, \quad x_0 = 2, \quad dx = 0.1$

In Exercises 31–36, write a differential formula that estimates the given change in volume or surface area.

31. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$.