c) The value of f at $x = 0$ is 0, and the value of f at $x = 3$ is 15. So on $[0, 3]$, f has an absolute minimum of -3.08 at $x = 1.15$, a local maximum of 0 at $x = 0$, and an absolute maximum of 15 at $x = 3$ (Fig. 4.12c).

Exercises 4.1

In Exercises 1–8, determine the critical points of the function.

1. $f(x)=x^3+x^2-8x+5$ **2.** $f(x)=x^3-2x^2-15x+2$

3. $F(x)=\sqrt[3]{x}, -1 \le x \le 8$ **4.** $F(x)=\sin x, -\frac{\pi}{2} \le x \le \frac{5\pi}{6}$

5. $g(x) = \sqrt{3+2x-x^2}$ **6.** $g(x) = \sqrt{5-4x-x^2}$

7. $h(x) = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$

8. $h(x) = \begin{cases} -\frac{x^2}{2}, & x < 0 \\ \frac{x^2}{2}, & 0 \le x < 2 \\ 2x-2, & x \ge 2 \end{cases}$

9. **a)** Draw a complete graph of each polynomial function. On a number line, mark the zeros of the polynomial together with the zeros of its derivative.

i) $y = x^2 + 8x + 15$ **ii)** $y = x^3 - 3x^2 + 4$

iii) $y = x^3 - 33x^2 + 216x$ **iv)** $y = -x^3 + 4x - 2$

On the basis of these four trials, make a conjecture about how the zeros of a polynomial and the zeros of its derivative are related. Do you believe the conjecture to be true? Give a convincing argument.

b) Use Rolle's Theorem to prove that between every two zeros of the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

there lies a zero of the polynomial

$$nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

10. The function

$$y = f(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ 0 & \text{if } x = 1, \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative at every point between 0 and 1 is $y' = 1$, and so y' is never zero between 0 and 1. Why doesn't that contradict Rolle's Theorem?

11. **a)** Draw a complete graph of $y = \sin x$. On a number line, mark its zeros together with the zeros of its derivative.

b) Use Rolle's Theorem to prove that between every two zeros of $y = \sin x$, there is a zero of $y = \cos x$.

12. **a)** Draw a complete graph of $y = (x^2 - 4x + 2)/(x + 1)$. On a number line, mark its zeros together with the zeros of its derivative.

b) Can you use Rolle's Theorem to prove the existence of each zero of the derivative of y?

13. Let $f(x) = |x^3 - 9x|$.

a) Determine a complete graph of f.

b) Does $f'(0)$ exist? Explain.

c) Does $f'(-3)$ exist? Explain.

d) Determine all local extrema of f.

e) Are parts (b)–(d) in conflict with Theorem 1? Explain.

f) Determine the intervals on which f is increasing and the intervals on which f is decreasing.

14. Let $g(x) = (x - 2)^{2/3}$.

a) Determine a complete graph of g.

b) Does $g'(2)$ exist? Explain.

c) Determine all local extrema of g.

d) Are parts (b) and (c) in conflict with Theorem 1? Explain.

e) Determine the intervals on which g is increasing and the intervals on which g is decreasing.

In Exercises 15–22, determine the local extrema of the function, the intervals on which the function is increasing and the intervals on which the function is decreasing, and the absolute maximum and absolute minimum of the function on the specified interval.

15. $f(x) = 5x - x^2$, $[0, 6]$

16. $f(x) = x^2 - x - 12$, $[-4, 4]$

17. $f(x) = \sqrt{x-4}$, $[4, 8]$

18. $f(x) = 4 - \sqrt{x+2}$, $[-2, 5]$

19. $y(x) = x^4 - 10x^2 + 9$, $[-3, 3]$

20. $g(x) = -x^4 + 5x^2 - 4$, $[-3, 3]$

21. $h(x) = x^3 - 2x - 2\cos x$, $[-5, 5]$

22. $h(x) = \sin x + 2\cos 2x$, $[0, 2\pi]$

The Mean Value Theorem

In Exercises 23–26, find the value(s) of c that satisfies the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals shown.

23. $f(x) = x^2 + 2x - 1, \quad 0 \le x \le 1$

24. $f(x) = x^{2/3}, \quad 0 \le x \le 1$

25. $f(x) = x + \dfrac{1}{x}, \quad \dfrac{1}{2} \le x \le 2$

26. $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$

27. *Speeding.* A trucker handed in a ticket at a toll booth showing that in 2 hours the truck had covered 159 mi on a toll road on which the speed limit was 65 mph. The trucker was cited for speeding. Why?

28. *Temperature change.* It took 20 sec for a thermometer to rise from 10°F to 212°F when it was taken from a freezer and placed in boiling water. Explain why somewhere along the way the mercury was rising at exactly 10.1°F/sec.

29. *Triremes.* Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots.

30. *Running the marathon.* A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice, the marathoner was running at exactly 11 mph.

31. Sketch the graph of a differentiable function with a local minimum value that is greater than one of its local maximum values.

32. Suppose that $y = f(x)$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = k$. Use Rolle's Theorem to prove there is at least one number c between a and b at which $f'(c) = 0$.

For the functions and intervals specified in Exercises 33–36, find the value(s) of c that satisfy the extension of Rolle's Theorem given in Exercise 32.

33. $f(x) = x \sin x, [-4, 4]$

34. $f(x) = x^2 \cos x, [-5, 5]$

35. $f(x) = x^2 - 2x, [-2, 4]$

36. $f(x) = 3x - x^2, [0.5, 2.5]$

37. Show that $y = 1/x$ decreases on any interval on which it is defined.

38. Show that $y = 1/x^2$ increases on any interval to the left of the origin and decreases on any interval to the right of the origin.

39. Suppose the derivative of a differentiable function $f(x)$ is never zero on the interval $0 \le x \le 1$. Show that $f(0) \ne f(1)$.

40. Show that for any numbers a and b

$$|\sin b - \sin a| \le |b - a|.$$

41. Suppose that f is differentiable for $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

42. Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a)$ and $f(b)$ have opposite signs and $f' \ne 0$ between a and b. Show that f has exactly one zero between a and b.

Use Exercise 42 to show analytically that the equations in Exercises 43-46 have exactly one solution in the given interval.

43. $x^4 + 3x + 1 = 0, \quad -2 \le x \le -1$

44. $-x^3 - 3x + 1 = 0, \quad 0 \le x \le 1$

45. $x - \dfrac{2}{x} = 0, \quad 1 \le x \le 3$

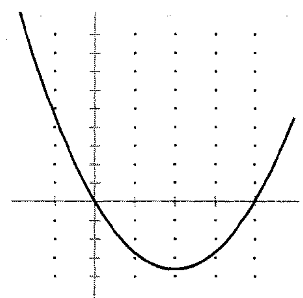
46. $2x - \cos x = 0, \quad -\pi \le x \le \pi$

47. Suppose that $f(0) = 3$ and that $f'(x) = 0$ for all x. Use the Mean Value Theorem to show that $f(x)$ must be 3 for all x.

48. Suppose that $f'(x) = 2$ and that $f(0) = 5$. Use the Mean Value Theorem to show that $f(x) = 2x + 5$ at every value of x.

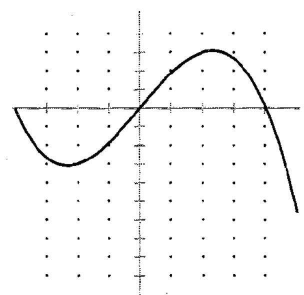
For the functions $y = f(x)$ graphed in Exercises 49 and 50, estimate where f' is (a) positive, (b) negative, and (c) zero.

49.



[–2, 5] by [–5, 10]

50.



[–2, 2.5] by [–10, 5]

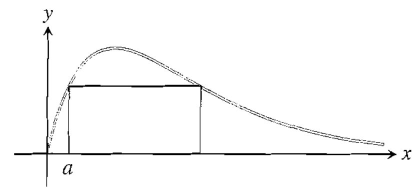
51. Find a function and real numbers a and b $(a < b)$ so that f is continuous on $[a, b]$, differentiable on (a, b), *not* differentiable on $[a, b]$, and has a tangent line at every point of $[a, b]$.

52. Suppose the graph of a differentiable function $y = f(x)$ passes through the point (1, 1). Sketch the graph if
a) $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$.
b) $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$.
c) $f'(x) > 0$ for $x \ne 1$.
d) $f'(x) < 0$ for $x \ne 1$.

53. Sketch one possible graph of $y = g(x)$ if $g(-2) = 1$ and $g'(x)$ is the function graphed in Exercise 49.

54. Sketch one possible graph of $y = g(x)$ if $g(-2) = 1$ and $g'(x)$ is the function graphed in Exercise 50.

55. *Challenge.* Consider the graph of $y = f(x) = xe^{-x}$ shown here. For each $a > 0$, think of the rectangle drawn in the first quadrant under the graph of f as suggested.



a) Determine a where the rectangle has area zero.
b) Compute the area of the rectangle for $a = 0.5, 0.8, 1.0, 1.2, 1.5$.
c) Determine a so that the rectangle has maximum area. What is the area? This may require nonstandard numerical techniques.

4.2 Predicting Hidden Behavior

Computer-drawn graphs of most functions that appear in this textbook are usually very reliable. In this section we will see how to use calculus to confirm completeness of graphs determined technologically and to predict behavior that is hidden from view on a computer graph. The verification that the graph really looks like what's on the screen and an analysis of any hidden behavior must come from calculus. The computer can only suggest what *might* be true.

We also introduce the concepts of concavity of graphs and of points where the sense of concavity changes, commonly called points of inflection of graphs. We will see that points of inflection are easy to locate by using a graphing utility even though they may be difficult to *see* in a viewing window because of the pixel nature of the graphs and the local straightness which shows if we ZOOM-IN.

The First Derivative

When we know that a function has a derivative at every point of an interval, we also know that it is continuous throughout the interval (Section 3.1) and that its graph over the interval is connected (Section 2.2). Thus, the graphs of $y = \sin x$ and $y = \cos x$ remain unbroken however far extended, as do the graphs of polynomials. The graphs of $y = \tan x$ and $y = 1/x^2$ are not connected only at points where the functions are undefined. On every interval that avoids these points, the functions are differentiable, so they are continuous and have connected graphs.

We gain additional information about the shape of a function's graph when we know where the function's first derivative is positive, negative, or zero. For, as we saw in Section 4.1, this tells us where the graph is rising, falling, or possibly has a horizontal tangent (Fig. 4.13.).