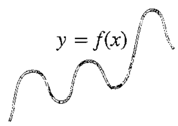
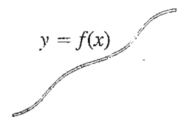
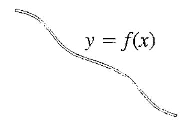
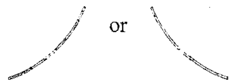
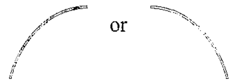
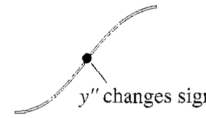
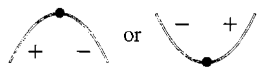
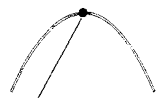
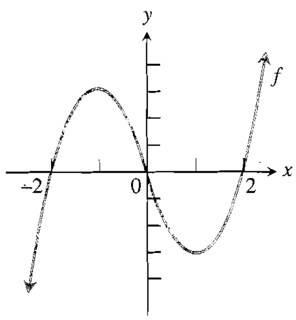
Summary: What Derivatives Confirm About Functions and Graphs

 Differentiable ⇒ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ graph rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ graph falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 Inflection point
 y' changes sign ⇒ graph has local maximum or minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

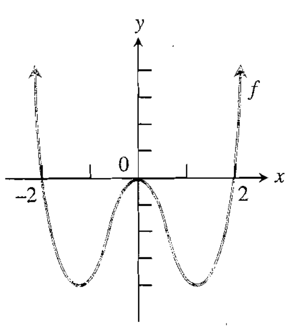
Exercises 4.2

For each function f graphed in Exercises 1 and 2, identify where the derivative f' is 0, positive, and negative.

1.

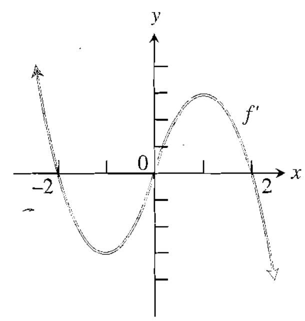


2.

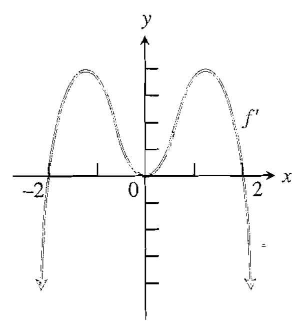


For each function f whose derivative is graphed in Exercises 3 and 4, identify the intervals on which the graph of f is rising or falling. Identify the local extreme values of f.

3.



4.



In Exercises 5–32, show a complete graph and identify the inflection points, local maximum and minimum values, and the intervals on which the graph is rising, falling, concave up, and concave down. We suggest you do exercises from the three groups in the order given.

Do Exercises 5–10 analytically, then support graphically.

5. $y = x^2 - x - 1$
6. $y = 4x^2 + 8x + 1$
7. $y = x^3 - 6x^2 + 9x + 1$
8. $y = -2x^3 + 6x^2 - 3$
9. $y = 2x^4 - 4x^2 + 1$
10. $y = x^4 - 2x^2$

Do Exercises 11–16 graphically, then confirm analytically.

11. $y = 2x^3 - 5x^2 + 4x + 10$

12. $y = 4x^3 + 21x^2 + 36x - 20$

13. $y = 3x^4 - x^2 - 10$

14. $y = 20 + 2x^2 - 9x^4$

15. $y = x + \sin x, \quad 0 \le x \le 2\pi$

16. $y = x - \sin x, \quad 0 \le x \le 2\pi$

Do Exercises 17–32 using a method of your choice.

17. $y = x^4 - 8x^2 + 4x + 2$

18. $y = -x^4 + 4x^3 - 4x + 1$

19. $y = -x^4 + 2x^2 - 3x - 2$

20. $y = 2x^4 - x^2 - 3x + 5$

21. $y = 2x^{1/5} + 3$
22. $y = 5 - x^{1/3}$

23. $y = 3x^{1/2} - 1$
24. $y = 2x^{1/4}$

25. $y = \dfrac{5}{1 + 2^{1-0.5x}}$
26. $y = \dfrac{-7}{1 + 3^{1-0.5x}}$

27. $f(x) = \begin{cases} x^2 + 1, & x \ge 0 \\ 3 - x^2, & x < 0 \end{cases}$

28. $f(x) = \begin{cases} 2 - x^2, & x \ge 1 \\ 2x, & x < 1 \end{cases}$

29. $y = x^{2/3}(3 - x)$
30. $y = x^{3/4}(5 - x)$

31. $y = x^{1/3}(x - 4)$
32. $y = x^{1/4}(x + 3)$

Analyze the motion of the particle moving on the x-axis with distance from the origin given by each function in Exercises 33–36. Support the analysis by graphing the parametric equations

$$x_1(t) = s(t), \quad y_1(t) = 2, \text{ and}$$

$$x_2(t) = s(t), \quad y_2(t) = t,$$

and using TRACE.

33. $x(t) = s(t) = t^2 - 4t + 3$
34. $x(t) = s(t) = 6 - 2t - t^2$
35. $x(t) = s(t) = t^3 - 3t + 3$
36. $x(t) = s(t) = 3t^2 - 2t^3$

In Exercises 37 and 38, the derivative of the function $y = f(x)$ is given. At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

37. $y' = (x - 1)^2(x - 2)$

38. $y' = (x - 1)^2(x - 2)(x - 4)$

Find the local maximum and minimum values of the functions in Exercises 39 and 40.

39. $y = x + \dfrac{1}{x}$
40. $y = \dfrac{x}{2} + \dfrac{1}{2x - 1}$

41. If $f(x)$ is a differentiable function and $f'(c) = 0$ at an interior point c of f's domain, must f have a local maximum or minimum at $x = c$? Explain.

42. If $f(x)$ is a twice-differentiable function and $f''(c) = 0$ at an interior point c of f's domain, must the graph of f have an inflection point at $x = c$? Explain.

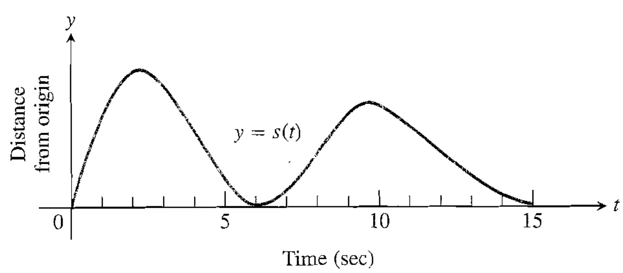
43. *Quadratic curves.* True or false? A quadratic curve $y = ax^2 + bx + c$ never has an inflection point. Explain.

44. *Cubic curves.* True or false? A cubic curve $y = ax^3 + bx^2 + cx + d$, $a \ne 0$, always has one inflection point. Explain.

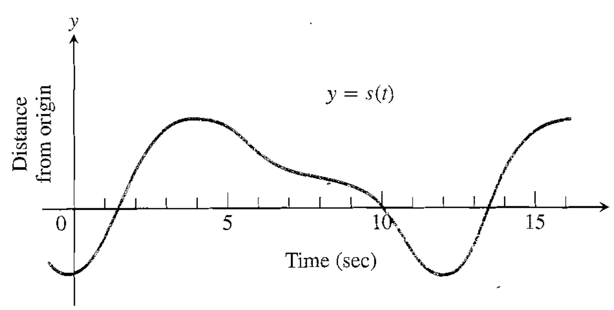
Velocity and Acceleration

Each graph in Exercises 45 and 46 is the graph of the position function $y = s(t)$ of a body moving back and forth on a coordinate line. At approximately what times is each body's (a) velocity equal to zero? (b) acceleration equal to zero?

45.



46.



47. Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

48. Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

49. Sketch a continuous curve $y = f(x)$ having the following characteristics:

$f(-2) = 8$,	$f'(2) = f'(-2) = 0$,
$f(0) = 4$,	$f'(x) < 0$ for $\|x\| < 2$,
$f(2) = 0$,	$f''(x) < 0$ for $x < 0$,
$f'(x) > 0$ for $\|x\| > 2$,	$f''(x) > 0$ for $x > 0$.

50. Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Curve
$x < 2$		Falling, concave up
2	1	Horizontal tangent
$2 < x < 4$		Rising, concave up
4	4	Inflection point
$4 < x < 6$		Rising, concave down
6	7	Horizontal tangent
$x > 6$		Falling, concave down

Linearizations at Inflection Points

Linearizations fit particularly well at points of inflection. You will see what we mean if you graph the pairs of functions in Exercises 51–54.

51. $f(x) = \sin x$ and its linearization $L(x) = x$ at $x = 0$.

52. $f(x) = \sin x$ and its linearization $L(x) = -x + \pi$ at $x = \pi$.

53. *Newton's serpentine.* $f(x) = 4x/(x^2 + 1)$ and its linearization $L(x) = 4x$ at $x = 0$.

54. *Newton's serpentine.* $f(x) = 4x/(x^2 + 1)$ and its linearization $L(x) = -x/2 + 3\sqrt{3}/2$ at the point $(\sqrt{3}, \sqrt{3})$.

55. Let $y_1 = f(x) = x^3 - 9x$.

a) Show that f has a point of inflection at (0, 0).

b) Determine m and b so that $y_2 = mx + b$ is the equation of the tangent line to the graph of f at $(a, f(a))$.

c) Let $a = 1$. GRAPH y_1 and y_2 simultaneously and ZOOM-IN on the point $(1, f(1))$ in the graph of f. Is the graph of y_1 above or below the graph of y_2? Explain.

d) Repeat part (c) with $a = 2, 3, -1, -2, -3$.

56. The function $f(x) = x^3 - 2x - 2\cos x$ has a point of inflection in $-1 < x < 0$.

a) Try to find the coordinates of this point of inflection by using ZOOM-IN. Explain why this procedure does not work.

b) Find the coordinates of the point of inflection by using $y = \text{NDER2}\, f$, and ZOOM-IN or SOLVE.

c) *Group Discussion.* Can you *confirm* part (b) analytically? Explain.

A ball is hit at an angle of elevation α and with initial velocity v_0. Neglecting air resistance, the position of the ball in Cartesian coordinates at time t is

$$(x(t), y(t)) = (v_0 t \cos\alpha,\ v_0 t \sin\alpha - 16t^2).$$

57. Use your grapher in parametric mode to simulate the motion of the ball hit with $v_0 = 90$ ft/sec and $\alpha = 40°, 50°, 60°, 75°, 80°, 85°$.

58. Determine formulas in terms of v_0 and α for the length of time the ball is in the air, the maximum height of the ball, and the range of the ball (the distance from the origin to the point of impact on level ground).

59. Determine the angle α so that the maximum height of the ball is equal to the range. Is this angle independent of v_0?

4.3 Polynomial Functions, Newton's Method, and Optimization

A complete graph has no important behavior hidden from view. In Section 1.1, we suggested what complete graphs of linear, quadratic, and cubic functions look like. Now we are equipped to establish what a complete graph of any polynomial function looks like. Because complete graphs also require that we know about the intercepts, we will look at a numerical method called Newton's method, or the Newton-Raphson method, for solving an equation $f(x) = 0$. We will close the section with applications involving *optimization*—finding values of x that give maximum or minimum values of $f(x)$ where f is a function that models the real situation.