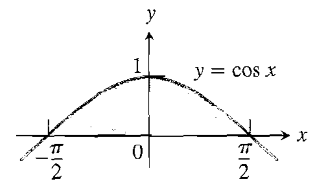


5.27 One arch of the curve $y = \cos x$. The area under it is 2.

EXPLORATION 2

Area Patterns in the Unit Square—Group Project

The efforts of a mathematician sometimes involve thoughtful, even wishful, guessing. To formulate and test some conjectures, we would like you to work together in groups of two or three.

1. GRAPH $y_1 = x^1$, $y_2 = x^2$, $y_3 = x^3$, and so on (as many as you want) in a square viewing window [0, 1] by [0, 1]. The area under $y_1 = x^1$ from 0 to 1 is 1/2 unit^2. Explain. With this area in mind and the graphs of y_i in view, make a conjecture about the explicit value of the area under each y_i curve over [0, 1]. Confirm or disprove your conjecture.
2. If you were able to confirm your conjecture in part 1, extend it to $i = 0$, -1, -2, and so forth. Then confirm or disprove, and explain. Extend it to rational exponents. Discuss your findings.

EXPLORATION BIT

In Exploration 2 of Section 5.2, we stated that $\int_0^\pi \sin x = 2$, or the area under the graph of $y = \sin x$ from 0 to π is 2. Now confirm this value in at least two different ways. Example 2 may give you a clue or two.

EXAMPLE 2

Find the area under one arch of the curve $y = \cos x$ (Fig. 5.27).

Solution

STEP 1: Find an antiderivative of $y = \cos x$:

$$F(x) = \sin x. \qquad \text{Simplest one}$$

STEP 2: Calculate $F\left(\frac{\pi}{2}\right) - F\left(-\frac{\pi}{2}\right)$:

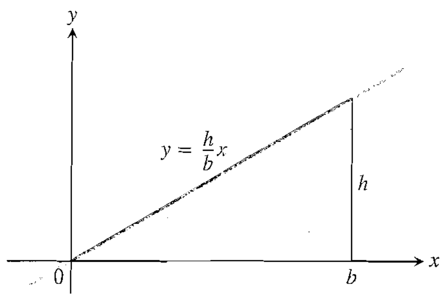
$$F\left(\frac{\pi}{2}\right) - F\left(-\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 1 - (-1) = 2.$$

The area is 2 square units.

Exercises 5.3

In Exercises 1–16, use your grapher or your prior knowledge of the graph to check that the function is nonnegative on the given interval. If it is, then use antiderivatives to find the area under the graph on the given interval. Support your result using NINT.

1. $y = x^2$, $[0, 2]$
2. $y = x^2$, $[-2, 1]$
3. $y = \sqrt{x}$, $[0, 4]$
4. $y = 1/x^2$, $[1/2, 2]$
5. $y = 2 - \sqrt{x}$, $[0, 4]$
6. $y = 1 - x^2$, $[-1, 1]$
7. $y = x^3 - 3x^2 + 4$, $[-1, 2]$
8. $y = 2x^4 - 4x^2 + 2$, $[-1, 1]$
9. $y = \cos x$, $[0, \pi/2]$
10. $y = 2\cos 2x$, $[0, \pi/4]$
11. $y = \sin 2x$, the interval under any one arch
12. $y = (1/2)\sin x$, the interval under any one arch
13. $y = \cos \pi x$, the interval under any one arch
14. $y = 1 + \sin x$, $[0, \pi]$
15. $y = \sec^2 x$, $[-\pi/4, \pi/3]$
16. $y = \sec x \tan x$, $[0, \pi/3]$
17. Let b be any positive number and n any rational number other than -1. Find a formula for the area under the curve $y = x^n$ from $x = 0$ to $x = b$.
18. Whenever we find a new way to calculate something, it is a good idea to be sure that the new and old ways agree on the objects to which they both apply. If you use an antiderivative to find the area of the following triangle, will you still get $A = (1/2)bh$? Try it and find out.



19. Suppose that f and g are continuous and that

$$\int_1^2 f(x)\,dx = -4,\ \int_1^5 f(x)\,dx = 6,\ \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find

a) $\int_2^2 g(x)\,dx$ b) $\int_5^1 g(x)\,dx$

c) $\int_1^2 3f(x)\,dx$ d) $\int_2^5 f(x)\,dx$

e) $\int_1^5 [f(x) - g(x)]\,dx$ f) $\int_1^5 [4f(x) - g(x)]\,dx.$

20. Suppose f and h are continuous and that

$$\int_1^9 f(x)\,dx = -1,\ \int_7^9 f(x)\,dx = 5,\ \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find

a) $\int_1^9 -2f(x)\,dx$ b) $\int_7^9 [f(x) + h(x)]\,dx$

c) $\int_7^9 [2f(x) - 3h(x)]\,dx$

d) $\int_9^1 f(x)\,dx$

e) $\int_1^7 f(x)\,dx$ f) $\int_9^7 [h(x) - f(x)]\,dx.$

21. Suppose that f is continuous and that

$$\int_1^2 f(x)\,dx = 5,\quad \int_1^3 f(z)\,dz = 2.$$

Find

a) $\int_1^2 f(u)\,du,$ b) $\int_2^1 f(t)\,dt,$ c) $\int_2^3 f(y)\,dy.$

22. Suppose that f is continuous and that

$$\int_0^3 f(x)\,dx = 3,\quad \int_0^4 f(z)\,dz = 7.$$

Find

a) $\int_0^3 f(t)\,dt,$ b) $\int_4^0 f(w)\,dw,$ c) $\int_3^4 f(y)\,dy.$

In Exercises 23–28, find the average value of f on $[a, b]$.

23. $f(x) = 2 - x^2$ on $[-3, 5]$

24. $f(x) = x^3 - 1$ on $[-1, 3]$

25. $f(x) = \sqrt{x}$ on $[0, 3]$ **26.** $f(x) = \cos^2 x$ on $[-\pi, \pi]$

27. $f(x) = x \sin x$ on $[0, 5]$ **28.** $f(x) = 2e^{-x^2}$ on $[1, 4]$

29. Use Rule 7 in Table 5.3 to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

30. *Continuation of Exercise 29.* Use Rule 7 in Table 5.3 to find upper and lower bounds for the values of

$$\int_0^{1/2} \frac{1}{1+x^2}\,dx \quad\text{and}\quad \int_{1/2}^1 \frac{1}{1+x^2}\,dx.$$

Then add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

Support with a NINT computation.

In Exercises 31–34, the value of $\int_a^b f(x)\,dx$ is given. Is there a point c in (a, b) such that $f(c)(b - a) = \int_a^b f(x)\,dx$? If so, find it. Support your result with a grapher by drawing the curve and the corresponding rectangle.

31. $\int_0^1 (x^3 + 1)\,dx = \frac{5}{4}$

32. $\int_0^3 (9 - x^2)\,dx = 18$

33. $\int_0^1 \frac{1}{x^2 + 1}\,dx = \frac{\pi}{4}$

34. $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\,dx = \frac{\pi}{3}$

35. The average value of $f(t)$ on $[-1, 2]$ is 5, and the average value of $f(t)$ on $[2, 7]$ is 3. Can you determine the average value of $f(t)$ on $[-1, 7]$? If so, what is it? Give reasons for your answers.

36. The average value of $g(s)$ on $[-5, -1]$ is 10, and the average value of $g(s)$ on $[-5, 10]$ is 50. Can you determine the average value of $g(s)$ on $[-1, 10]$? If so, what is it? Give reasons for your answers.

37. Suppose that f is continuous and that

$$\int_1^2 f(x)\,dx = 4.$$

Show that $f(x) = 4$ at least once on the interval $[1, 2]$.

38. Show that the value of

$$\int_0^1 \sin^2 x\,dx$$

cannot possibly be 2.