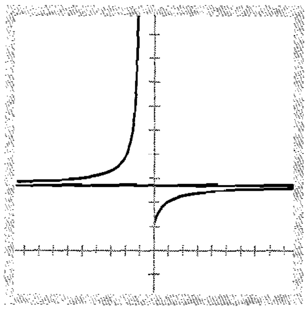
AN ALTERNATIVE DEFINITION OF *e*

We could have defined e to be $\lim_{x\to\infty}(1+1/x)^x$. However, the proof that this limit exists would have to be different from the logarithm-based proof in Example 6.



[−10, 10] by [−2, 10]

7.30 Graphs of $y_1 = f(x) = (1+\frac{1}{x})^x$ and $y_2 = e$.

In the exercises, we ask you to show that the graph of f is complete.

EXAMPLE 6

Let $f(x) = (1 + 1/x)^x$. Show that $\lim_{x\to\pm\infty} f(x) = e$, that is, $y = e$ is a horizontal asymptote of $f(x)$. Draw a complete graph of f.

Solution The function f is defined outside of $[-1, 0]$. As $x \to \pm\infty$, $(1 + 1/x) \to 1$. Therefore, $f(x)$ approaches the indeterminate form 1^∞, so we take logarithms.

$$\ln f(x) = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

The latter expression gives the indeterminate form 0/0, so we apply l'Hôpital's Rule:

$$\lim_{x\to\pm\infty}\ln f(x) = \lim_{x\to\pm\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \frac{0}{0}$$

$$= \lim_{x\to\pm\infty}\left(\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\pm\infty}\frac{1}{1+\frac{1}{x}} = 1.$$

Therefore,

$$\lim_{x\to\pm\infty} f(x) = e^1 = e. \qquad \text{Eq. (1) with } L = 1$$

A complete graph is shown in Fig. 7.30 ■

Exercises 7.5

Find the limits in Exercises 1–8. Use l'Hôpital's Rule when the form is indeterminate. Support your answer graphically.

1. $\lim_{x\to 2}\frac{x-2}{x^2-4}$
2. $\lim_{x\to 2}\frac{x^2-4}{x-2}$
3. $\lim_{x\to 1}\frac{x^3-1}{4x^3-x-3}$
4. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$
5. $\lim_{t\to 0}\frac{\sin t^2}{t}$
6. $\lim_{x\to 0}\frac{\sin 5x}{x}$
7. $\lim_{x\to\infty}\frac{3x^2-1}{2x^2-x+1}$
8. $\lim_{t\to\infty}\frac{6t+5}{3t-8}$

Use graphs to find the limits in Exercises 9–20. Confirm your answers analytically. Use l'Hôpital's Rule when the form is indeterminate.

9. $\lim_{x\to\pi/2}\frac{2x-\pi}{\cos x}$
10. $\lim_{x\to 0}\frac{(1/2)^x-1}{x}$
11. $\lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1}$
12. $\lim_{t\to 0}\frac{\cos t-1}{t^2}$
13. $\lim_{x\to\pi/2}\frac{1-\sin x}{1+\cos 2x}$
14. $\lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)\tan x$
15. $\lim_{x\to 0^+}\frac{2x}{x+7\sqrt{x}}$
16. $\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}$
17. $\lim_{t\to 0}\frac{10(\sin t-t)}{t^3}$
18. $\lim_{x\to 0}\frac{x(1-\cos x)}{x-\sin x}$
19. $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$
20. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)$

Use the technique of Example 6 to find the limits in Exercises 21–26 analytically, and support your answer graphically.

21. $\lim_{x\to 0^+} x^{(1/\ln x)}$
22. $\lim_{x\to 0^+} x^{1/x}$
23. $\lim_{x\to 0^+}(e^x+x)^{1/x}$
24. $\lim_{x\to 1} x^{1/(x-1)}$
25. $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^x$
26. $\lim_{x\to 0}\left(\ln\left|\frac{1}{x}\right|\right)^x$

Is it possible to extend the functions in Exercises 27–30 so that they are continuous at $x = 0$? If so, explain each extension.