Exercises 2.6

In Exercises 1–4, sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find the largest value of $\delta > 0$ such that $|x - x_0| < \delta$ implies that $a < x < b$.

1. $a = 1$, $b = 7$, $x_0 = 5$ **2.** $a = 1$, $b = 7$, $x_0 = 2$

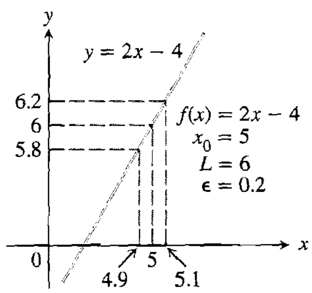
3. $a = -7/2$, $b = -1/2$, $x_0 = -3$

4. $a = -7/2$, $b = -1/2$, $x_0 = -3/2$

Use the graphs in Exercises 5–10 to find a $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

5.



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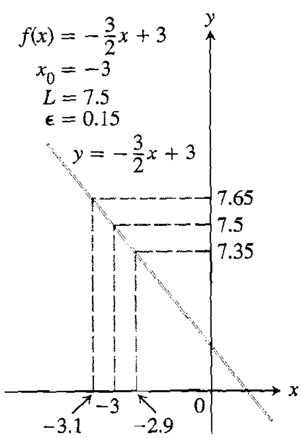

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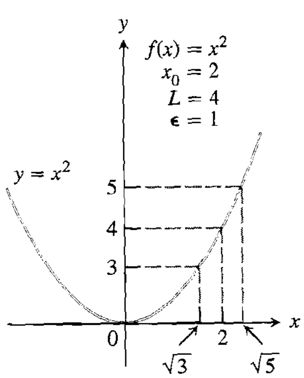

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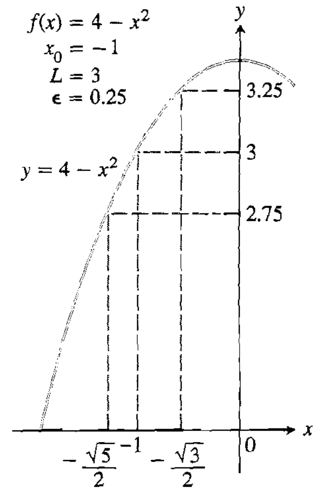

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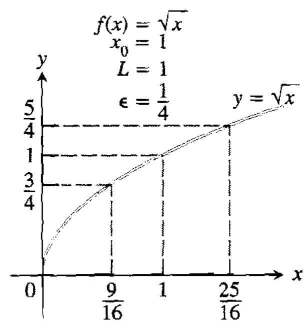

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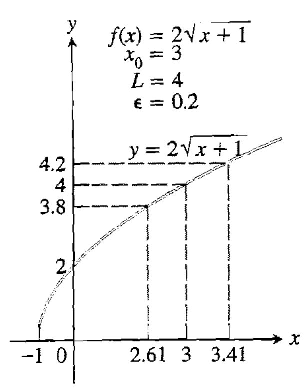


NOT TO SCALE

Each of Exercises 11–18 gives a function $f(x)$, a point x_0, and a positive number ϵ. Find $L = \lim_{x \to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

11. $f(x) = 2x + 3, x_0 = 1, \epsilon = 0.01$

12. $f(x) = 3 - 2x, x_0 = 3, \epsilon = 0.02$

13. $f(x) = \dfrac{x^2 - 4}{x - 2}, x_0 = 2, \epsilon = 0.05$

14. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, x_0 = -5, \epsilon = 0.05$

15. $f(x) = \sqrt{x - 7}, x_0 = 11, \epsilon = 0.01$

16. $f(x) = \sqrt{1 - 5x}, x_0 = -3, \epsilon = 0.5$

17. $f(x) = 4/x, x_0 = 2, \epsilon = 0.4$

18. $f(x) = 4/x, x_0 = 1/2, \epsilon = 0.04$

In Exercises 19 and 20, find the largest $\delta > 0$ such that for all x,

$$0 < |x - 4| < \delta \quad \Rightarrow \quad |f(x) - 5| < \epsilon.$$

19. $f(x) = 9 - x$; $\epsilon = 0.01, 0.001, 0.0001$, arbitrary $\epsilon > 0$

20. $f(x) = 3x - 7$; $\epsilon = 0.003, 0.0003$, arbitrary $\epsilon > 0$

In Exercises 21–24, evaluate and confirm the limits using the definitions of limits involving infinity.

21. $\lim_{x \to \infty} \dfrac{x + 2}{x + 1}$

22. $\lim_{x \to \infty} \dfrac{x^2}{2x^2 - 1}$

23. $\lim_{x \to -1^+} \dfrac{x + 2}{x + 1}$

24. $\lim\limits_{x \to (\sqrt{2}/2)^+} \dfrac{x^2}{2x^2 - 1}$

The functions in Exercises 25–32 are locally straight. Let $L = \lim_{x \to x_0} f(x)$, $\epsilon > 0$, and estimate $\delta > 0$ so that $|x - x_0| < \delta$ implies that $|f(x) - L| < \epsilon$. You may assume that ϵ is small, say $\epsilon < 0.01$.

25. $f(x) = \sin x, x_0 = 1$

26. $f(x) = \tan x, x_0 = 1$

27. $f(x) = \cos x, x_0 = 1$

28. $f(x) = \sec x, x_0 = 4$

29. $f(x) = x^3 - 4x, x_0 = 0.5$

30. $f(x) = 9x - x^3, x_0 = 2.5$

31. $f(x) = \dfrac{x}{x^2 - 4}, x_0 = -1$

32. $f(x) = \dfrac{2x}{5 - x^2}, x_0 = -1$

33. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x - 5} < \epsilon$. What limit is being verified?

34. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4 - x} < \epsilon$. What limit is being verified?

35. Graph the function

$$f(x) = \begin{cases} 4 - 2x, & x < 1, \\ 6x - 4, & x \geq 1. \end{cases}$$

Then, given $\epsilon > 0$, find the largest δ for which $f(x)$ lies between $y = 2 - \epsilon$ and $y = 2 + \epsilon$ for x in the interval $I = (1 - \delta, 1 + \delta)$.

36. Let $f(x) = |x - 5|/(x - 5)$. Find the set of x-values for which

$$1 - \epsilon < f(x) < 1 + \epsilon, \qquad \text{for } \epsilon = 4, 2, 1, \text{ and } 1/2.$$

37. Define what it means to say that $\lim_{x \to 2} f(x) = 5$.

38. Define what it means to say that $\lim_{x \to 0} g(x) = k$.

39. Suppose $0 < \epsilon < 4$. Find the largest $\delta > 0$ with the property that $|x - 2| < \delta$ implies that $|x^2 - 4| < \epsilon$. What limit is being verified? What happens to δ as ϵ decreases toward 0? Graph δ as a function of ϵ.

40. Suppose $0 < \epsilon < 1$. Find the largest $\delta > 0$ with the property that $|x - 3| < \delta$ implies that $\left| \dfrac{2}{x - 1} - 1 \right| < \epsilon$. What limit is being verified? What happens to δ as ϵ decreases toward 0? Graph δ as a function of ϵ.

41. Let $f(x) = 1/x$. Show that $|x - 0.5| < 0.00248$ implies that $|f(x) - 2| < 0.01$. (See Example 3.)

42. Let $0 < \epsilon < 2$.

a) Use graphs to support $\dfrac{1}{2 + \epsilon} < \dfrac{1}{2} < \dfrac{1}{2 - \epsilon}$.

b) Show that for $\epsilon > 2$, the inequality statement in part (a) does not hold. Which part fails?

43. Let $f(x) = 1/x$ and

$$\delta = \begin{cases} \dfrac{\epsilon}{2(2 + \epsilon)}, & \text{if } \epsilon < 2, \\ 1/4, & \text{if } \epsilon \geq 2. \end{cases}$$

Show that $|x - 0.5| < \delta$ implies that $|f(x) - 2| < \epsilon$. (See Example 4 and Exercise 42.)

44. Let

$$f(x) = \begin{cases} x^3 + 1.001, & x \geq 0, \\ x^3 + 0.009, & x < 0. \end{cases}$$

a) Draw a complete graph of f.

b) Find $\lim_{x \to 0} f(x)$ if it exists.

c) If the limit in part (b) does not exist, show how the implication

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

fails.

45. Suppose

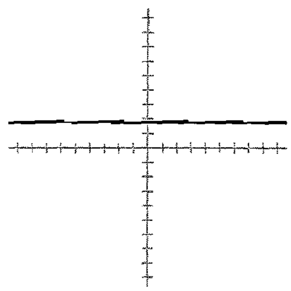
$$f(x) = \begin{cases} x, & \text{if } x = 1/n \text{ for every natural number } n, \\ 0, & \text{otherwise.} \end{cases}$$

a) Graph $f(x)$.

b) Does $\lim_{x \to 0} f(x)$ exist? If so, find the limit. Explain.

46. Note that $\lim_{x \to -1} |x + 1|/(x + 1)$ does not exist. Prove the limit is not 1 with an ϵ-δ argument. That is, show that there is an $\epsilon > 0$ such that for each $\delta > 0$ there is always an x_0 with $0 < |x_0 + 1| < \delta$ but $||x + 1|/(x + 1) - 1| \geq \epsilon$. (*Hint:* Choose a value for δ, say $\delta = 2$. Then find an ϵ that makes the last inequality hold. Finally, complete the argument for any arbitrary δ.)

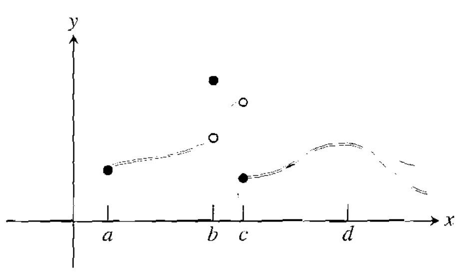
1. You have been asked to calculate the limit of a function $f(x)$ as x approaches a finite number c. What theorems are available for calculating the limit? Give examples to show how the theorems are used.
2. What is the relation between one-sided and two-sided limits? How is this relation sometimes used to calculate a limit or to prove that a limit does not exist? Give examples.
3. You used a graphing calculator to estimate $\lim_{\theta\to 0}(\sin\theta)/\theta$ and found the graph of $f(\theta) = (\sin\theta)/\theta$ to be the horizontal line $y = 0.0174532925 \approx \pi/180$ in this figure. Explain what you did wrong.



$[-0.1, 0.1]$ by $[-0.1, 0.1]$

4. What is the procedure for finding the limit of a rational function of x as x approaches $\pm\infty$? When is the limit zero? Finite and different from zero? Infinite?
5. What test can you apply to find out whether a function $y = f(x)$ is continuous at point $x = c$? Give examples of functions that are continuous at $x = 0$. Give examples of functions that fail to be continuous at $x = 0$ for various reasons. (They don't have to be examples from the book. You may make up your own.)
6. What can be said about the continuity of polynomial functions and rational functions?
7. What can be said about the continuity of composites of continuous functions?
8. What are the important theorems about continuous functions? Can functions that are not continuous be expected to have the properties guaranteed by these theorems? Give examples.
9. What are the formal definitions of (two-sided) limit, right-hand limit, and left-hand limit?
10. Discuss how a grapher can be used to *help* determine the limit of a function. Explain why this is not conclusive.
11. Define horizontal and vertical asymptote.
12. Give an example of a function with a removable discontinuity.
13. Define end behavior and end behavior model for a function.
14. Define end behavior asymptote for a rational function.
15. How are absolute values used to describe intervals of real numbers?
16. Show by example how absolute values are used to control function values.
17. State the formal definitions of limits of infinity and limits at infinity.

In Exercises 1–6, decide whether the limits exist on the basis of the graph of $y = f(x)$ shown. The domain of f is the set of real numbers.



1. $\lim_{x\to d} f(x)$
2. $\lim_{x\to c^+} f(x)$
3. $\lim_{x\to c^-} f(x)$
4. $\lim_{x\to c} f(x)$
5. $\lim_{x\to b} f(x)$
6. $\lim_{x\to a} f(x)$

In Exercises 7–10, decide whether $f(x)$ whose graph is shown above is continuous at the following points.

7. $x = a$
8. $x = b$
9. $x = c$
10. $x = d$

Find the limits in Exercises 11–30. Some limits may not exist.

11. $\lim_{x\to -2} x^2(x+1)$
12. $\lim_{x\to 3}(x+2)(x-5)$
13. $\lim_{x\to 3} \dfrac{x-3}{x^2}$
14. $\lim_{x\to -1} \dfrac{x^2+1}{3x^2-2x+5}$
15. $\lim_{x\to -2} \left(\dfrac{x}{x+1}\right)\left(\dfrac{3x+5}{x^2+x}\right)$

16. $\lim_{x\to 1}\left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

17. $\lim_{x\to 4}\sqrt{1-2x}$

18. $\lim_{x\to 5}\sqrt[4]{9-x^2}$

19. $\lim_{x\to 1}\frac{x^2-1}{x-1}$

20. $\lim_{x\to -5}\frac{x^2+3x-10}{x+5}$

21. $\lim_{x\to 2}\frac{x-2}{x^2+x-6}$

22. $\lim_{x\to 1}\frac{x^2-2x+1}{x^3-2x^2+x}$

23. $\lim_{x\to 0}\frac{(1+x)(2+x)-2}{x}$

24. $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

25. $\lim_{x\to\infty}\frac{2x+3}{5x+7}$

26. $\lim_{x\to -\infty}\frac{2x^2+3}{5x^2+7}$

27. $\lim_{x\to -\infty}\frac{x^2-4x+8}{3x^3}$

28. $\lim_{x\to\infty}\frac{1}{x^2-7x+1}$

29. $\lim_{x\to -\infty}\frac{x^2-7x}{x+1}$

30. $\lim_{x\to\infty}\frac{x^4+x^3}{12x^3+128}$

Find the limits in Exercises 31–34.

31. $\lim_{x\to 3^+}\frac{1}{x-3}$

32. $\lim_{x\to 3^-}\frac{1}{x-3}$

33. $\lim_{x\to 0^+}\frac{1}{x^2}$

34. $\lim_{x\to 0^-}\frac{1}{|x|}$

Find the limits in Exercises 35–38.

35. $\lim_{x\to 0}\frac{\sin 2x}{4x}$

36. $\lim_{x\to 0}\frac{x+\sin x}{x}$

37. $\lim_{x\to 0}\frac{\sin^3 2x}{x^3}$

38. $\lim_{x\to 0}\frac{2\csc 5x}{\csc 3x}$

39. Let $f(x)=\frac{\sec 2x\csc 9x}{\cot 7x}$.

a) Estimate $\lim_{x\to 0}f(x)$ graphically.

b) Compare the function values of f near 0 to the limit found in part (a).

c) Describe (informally, no proof required) the behavior of f near 0. (*Hint:* Restrict x to $[-0.2, 0.2]$.)

d) Find the exact value of $\lim_{x\to 0}f(x)$ algebraically.

40. Let $f(x)=\frac{\csc 5x\sec 8x}{\csc 8x\sec 3x}$.

a) Estimate $\lim_{x\to 0}f(x)$ graphically.

b) Compare the function values of f near 0 to the limit found in part (a).

c) Describe (informally, no proof required) the behavior of f near 0. (*Hint:* Restrict x to $[-0.2, 0.2]$.)

d) Find the exact value of $\lim_{x\to 0}f(x)$ algebraically.

In Exercises 41 and 42 find the limits.

41. **a)** $\lim_{x\to -2^+}\frac{x+3}{x+2}$ **b)** $\lim_{x\to -2^-}\frac{x+3}{x+2}$

42. **a)** $\lim_{x\to 2^+}\frac{x-1}{x^2(x-2)}$ **b)** $\lim_{x\to 2^-}\frac{x-1}{x^2(x-2)}$

c) $\lim_{x\to 0^+}\frac{x-1}{x^2(x-2)}$ **d)** $\lim_{x\to 0^-}\frac{x-1}{x^2(x-2)}$

43. Let

$$f(x)=\begin{cases}1, & x\le -1,\\ -x, & -1<x<0,\\ 1, & x=0,\\ -x, & 0<x<1,\\ 1, & x\ge 1.\end{cases}$$

a) Determine a complete graph of f.

b) Find the right-hand and left-hand limits of f at -1, 0, and 1.

c) Does f have a limit as x approaches -1? 0? 1? If so, what is it? If not, why not?

d) At which of the points $x=-1, 0, 1$, if any, is f continuous?

44. Repeat Exercise 43 for the function

$$f(x)=\begin{cases}0, & x\le -1,\\ |2x|, & -1<x<1,\\ 0, & x=1,\\ 1, & x>1.\end{cases}$$

45. Let $f(x)=\begin{cases}|x^3-4x|, & x<1,\\ x^2-2x-2, & x\ge 1.\end{cases}$

a) Determine a complete graph of f.

b) Find the right-hand and left-hand limits of f at 1.

c) Does f have a limit as x approaches 1? If so, what is it? If not, why not?

d) At what points is f continuous? Why?

e) At what points is f not continuous? Why?

46. Let $f(x)=\begin{cases}1-\sqrt{3-2x}, & x<3/2,\\ 1+\sqrt{2x-3}, & x\ge 3/2.\end{cases}$

a) Determine a complete graph of f.

b) Find the right-hand and left-hand limits of f at 3/2.

c) Does f have a limit as x approaches 3/2? If so, what is it? If not, why not?

d) At what points is f continuous? Why?

e) At what points is f not continuous? Why?

47. Let $f(x)=\begin{cases}-x, & x<1,\\ x-1, & x>1.\end{cases}$

a) Graph f.

b) Find the right-hand and left-hand limits of f at $x=1$.

c) What value, if any, should be assigned to $f(1)$ to make f continuous at $x=1$?

48. Repeat Exercise 47 for the function

$$f(x)=\begin{cases}3x^2, & x<1,\\ 4-x^2, & x>1.\end{cases}$$

Find the points, if any, at which the functions in Exercises 49 and 50 are *not* continuous.

49. $f(x)=\frac{x+1}{4-x^2}$

50. $f(x)=\sqrt[3]{3x+2}$

Find the end behavior asymptotes in Exercises 51–54 algebraically and support graphically.

51. $f(x)=\frac{2x+1}{x^2-2x+1}$

52. $g(x)=\frac{2x^2+5x-1}{x^2+2x}$

53. $h(x) = \dfrac{x^3 - 4x^2 + 3x + 3}{x - 3}$

54. $T(x) = \dfrac{x^4 - 3x^2 + x - 1}{x^3 - x + 1}$

55. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to c} f(x) = -7$ and $\lim_{x\to c} g(x) = 0$. Find the limit as $x \to c$ of the following functions.

a) $3f(x)$ b) $(f(x))^2$ c) $f(x) \cdot g(x)$

d) $\dfrac{f(x)}{g(x) - 7}$ e) $\cos(g(x))$ f) $|f(x)|$

56. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to 0} f(x) = \frac{1}{2}$ and $\lim_{x\to 0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.

a) $-g(x)$ b) $g(x) \cdot f(x)$ c) $f(x) + g(x)$

d) $1/f(x)$ e) $x + f(x)$ f) $\dfrac{f(x) \cdot \sin x}{x}$

57. Use the inequality

$$0 \le \left|\sqrt{x} \sin \frac{1}{x}\right| \le \sqrt{x}$$

to find $\lim_{x\to 0} \sqrt{x} \sin(1/x)$.

58. Use the inequality

$$0 \le \left|x^2 \sin \frac{1}{x}\right| \le x^2$$

to find $\lim_{x\to 0} x^2 \sin(1/x)$.

Given that $\lim_{x\to\infty} (\sin x)/x = \lim_{x\to\infty} (\cos x)/x = 0$, find the limits in Exercises 59 and 60.

59. $\displaystyle\lim_{x\to\infty} \frac{x + \sin x}{x}$ 60. $\displaystyle\lim_{x\to\infty} \frac{x + \sin x}{x + \cos x}$

Use the Sandwich Theorem to find the limits in Exercises 61 and 62.

61. $\displaystyle\lim_{x\to\infty} \left(\frac{\sin x}{\sqrt{x}}\right)$ 62. $\displaystyle\lim_{x\to\infty} \frac{\cos x}{\sqrt{x}}$

63. Let $f(x) = \begin{cases} \dfrac{x^2 + 2x - 15}{x - 3}, & x \ne 3, \\ k, & x = 3. \end{cases}$
What value, if any, should be assigned to k to make f continuous at $x = 3$?

64. With the help of a graphing utility, determine $\lim_{x\to 0^+} x^x$. Why must this limit be approached from the right-hand side?

65. Study $f(x) = (2^x + 3^x)^{1/x}$ near $x = 0$ to determine each limit.

a) $\lim_{x\to 0^-} f(x)$

b) $\lim_{x\to 0^+} f(x)$

c) Does $\lim_{x\to 0} f(x)$ exist? Explain.

66. Let $f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \ne 0, \\ k, & x = 0. \end{cases}$
What value, if any, should be assigned to k to make f continuous at $x = 0$?

67. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$, even though the function is continuous on this interval. Does this contradict the Max-Min Theorem for continuous functions? Why?

68. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Note that the interval is not closed. Is this consistent with the Max-Min Theorem for continuous functions? Why?

69. True or false? If $y = f(x)$ is continuous, with $f(1) = 0$ and $f(2) = 3$, then f takes on the value 2.5 at some point between $x = 1$ and $x = 2$. Explain.

70. Show that there is at least one value of x for which $x + \cos x = 0$.

71. How do you know (for sure) that $x + \log x = 0$ has at least one solution? Explain.

The Definition of Limit

72. Define what it means to say that

$$\lim_{x\to 1} f(x) = 3.$$

73. Define what it means to say that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Wrong descriptions of limit. Show by example that the statements in Exercises 74 and 75 are wrong.

74. The number L is the limit of $f(x)$ as x approaches x_0 if $f(x)$ gets closer to L as x approaches x_0.

75. The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there is a value of x for which $|f(x) - L| < \epsilon$.

Each of Exercises 76–81 gives a function $y = f(x)$, a number E, and a target value y_0. In each case, find an interval of x-values for which $y = f(x)$ lies within E units of y_0. (Recall our agreement to try to choose the interval as large as possible.) Then find an interval which can be described with an absolute value inequality of the form $|x - x_0| < D$.

	$f(x)$	E	y_0
76.	$\sqrt{x+2}$	1	4
77.	$\sqrt{\dfrac{x+1}{2}}$	1/2	1
78.	$\dfrac{x-1}{x-3}$	0.1	2
79.	$\dfrac{x-1}{x-3}$	0.1	-2
80.	$x^3 - 4x$	0.1	4
81.	$x^3 - 4x$	0.1	$1 (-1 < x_0 < 0)$

82. The function $f(x) = 2x - 3$ is continuous at $x = 2$. Given a positive number ϵ, how small must δ be for $|x - 2| < \delta$ to imply that $|f(x) - 1| < \epsilon$?

83. The function $f(x) = |x|$ is continuous at $x = 0$. Given a positive number ϵ, how small must δ be for $|x - 0| < \delta$ to imply that $|f(x) - 0| < \epsilon$?

In Exercises 84 and 85, evaluate and confirm the limits using the definitions of limits involving infinity.

84. $\lim_{x\to\infty} \dfrac{1-2x}{3x-1}$ **85.** $\lim\limits_{x\to(1/3)^+} \dfrac{1-2x}{3x-1}$

Each of Exercises 86–89 gives a function $f(x)$, a point x_0, and a positive number ϵ. Find $L = \lim_{x\to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

86. $f(x) = 5x - 10$, $x_0 = 3$, $\epsilon = 0.05$

87. $f(x) = 5x - 10$, $x_0 = 2$, $\epsilon = 0.05$

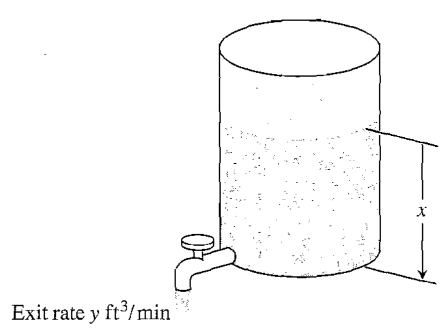
88. $f(x) = \sqrt{x-5}$, $x_0 = 9$, $\epsilon = 1$

89. $f(x) = \sqrt{2x-3}$, $x_0 = 2$, $\epsilon = 1/2$

The functions in Exercises in 90 and 91 are locally straight at x_0. Let $L = \lim_{x\to 0} f(x)$, $\epsilon > 0$, and estimate $\delta > 0$ so that $|x - x_0| < \delta$ implies that $|f(x) - L| < \epsilon$. You may assume that ϵ is small, say $\epsilon < 0.01$.

90. $f(x) = \dfrac{x^2 - x}{x+2}$, $x_0 = 5$ **91.** $f(x) = \dfrac{x-1}{x^2+3x}$, $x_0 = -5$

92. *Controlling the flow from a draining tank.* Torricelli's Law says that if you drain a tank like the one shown, the rate y at which the water runs out is a constant times the square root of the water's depth. As the tank drains, x decreases, and so does y, but y decreases less rapidly than x. The value of the constant depends on the size of the exit valve.



Suppose that for the tank in question, $y = \sqrt{x}/2$. You are trying to maintain a constant exit rate of $y_0 = 1$ ft^3/min by refilling the tank with a hose from time to time. How deep must you keep the water to hold the rate to within 0.2 ft^3/min of $y_0 = 1$? Within 0.1 ft^3/min of $y_0 = 1$? In other words, in what interval must you keep x to hold y within 0.2 (or 0.1) units of $y_0 = 1$?

Remark: What if we want to know how long it will take the tank to drain if we do not refill it? We cannot answer such a question with the usual equation Time = Amount/Rate, because the rate changes as the tank drains. We could always open the valve, sit down with a watch, and wait; but with a large tank or a reservoir, that might take hours or even days. With calculus, we will be able to find the answer in just a minute or two, as you will see if you do Exercise 47 in Section 4.7.

93. *Dimension changes in equipment.* As you probably know, most metals expand when heated and contract when cooled, and people sometimes have to take this into account in their work. Boston and Maine Railroad crews try to lay track at temperatures as close to 65°F as they can so that the track won't expand too much in the summer or shrink too much in the winter. Surveyors have to correct their measurements for temperature when they use steel measuring tapes.

The dimensions of a piece of laboratory equipment are often so critical that the machine shop in which it is made has to be held at the same temperature as the laboratory where a part is to be installed. And once the piece is installed, the laboratory must continue to be held at that temperature.

A typical aluminum bar that is 10 cm wide at 70°F will be

$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t. As t rises above 70, the bar's width increases; as t falls below 70, the bar's width decreases.

Suppose you had a bar like this made for a gravity-wave detector you were building. You need the width of the bar to stay within 0.0005 cm of the ideal 10 cm. How close to 70°F must you maintain the temperature of your laboratory to achieve this? In other words, how close to $t_0 = 70$ must you keep t to be sure that y lies within 0.0005 of $y_0 = 10$?

94. The equation $1000x^2 + x - 10^{-15} = 0$ has meaning in chemistry. (See *The Mathematics Teacher*, Vol. 85(6), p. 462.) Confirm that $x = 0$ is not a solution. Find the solution, both exactly and approximately. Explain your results, and discuss the error in your approximation.